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# On Existence and Scattering with Minimal Regularity for Semilinear Wave Equations*

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We prove existence and scattering results for semilinear wave equations with low regularity data. We also determine the minimal regularity that is needed to ensure local existence and well-posedness, and we give counterexamples to well-posedness. More specifically, we show that equations of the type $\Box u = |u|^p$, with initial data $(u, u_t)$ in $\dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)$, have a local solution if $\gamma \geqslant \gamma(p, n)$, and we construct counterexamples if $\gamma < \gamma(p, n)$. The existence results rely on mixed-norm space-time estimates of Strichartz-type. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

Let us consider the semilinear Cauchy problem

$$\begin{cases} \Box u = F_\kappa(u) \\ u(0, x) = f(x) \in \dot{H}^\gamma(\mathbb{R}^n), \qquad \partial_t u(0, x) = g(x) \in \dot{H}^{\gamma-1}(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where, for a given $\kappa > 1$, $F_\kappa$ is assumed to be a $C^1$ function satisfying

$$|F_\kappa(u)| \leqslant C\,|u|^\kappa, \qquad C^{-1}\,|F_\kappa(u)| \leqslant |uF'_\kappa(u)| \leqslant C\,|F_\kappa(u)|. \tag{1.2}$$

The main goal of this paper is to find the minimal $\gamma$, depending on $\kappa$ and $n \geqslant 2$, such that the conditions on $f$ and $g$ in (1.1) are enough to ensure that there is a (weak) solution of (1.1) satisfying

$$(u, \partial_t u) \in C_b([0, T_*];\ \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)) \tag{1.3}$$

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for some $0 < T_* \leqslant \infty$. Here $\dot{H}^\gamma$ denotes the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^\gamma} = \| \, |D_x|^\gamma f\|_{L^2}$ and $|D_x| = \sqrt{-\Delta_x}$, and $C_b$ denotes the space of bounded continuous functions.

It has been known for some time that non-trivial space-time estimates for the wave equation lead to good existence and scattering theorems for semilinear equations. This goes back, among other places, to pioneering work of Segal, Strauss and Strichartz. In the last few years there has been a lot of work in proving estimates for Fourier integrals arising in studying the wave equation. The proofs of our existence results use mixed space-time norm estimates similar to these. These estimates give sharp existence results and we also construct counterexamples showing that they are sharp. In addition, we shall show that, for a natural range of $\kappa$, there is asymptotic completeness and scattering for small-amplitude solutions of (1.1) with minimal regularity.

In order for the right side of (1.1) to be well-defined in the sense of distributions it suffices to have $\gamma \geqslant n(1/2 - 1/\kappa)$. However, it is not hard to see that this is not enough for existence, due to nonlinear effects such as blow-up. In fact, just by scaling one can get a simple counterexample to well-posedness if $\gamma < n/2 - 2/(\kappa - 1)$ (compare Ponce and Sideris [24]). This scaling argument relies on the fact that if $u$ solves

$$\Box u = |u|^\kappa, \qquad u(0, x) = f(x), \qquad \partial_t u(0, x) = g(x) \tag{1.4}$$

then $u_\varepsilon = \varepsilon^{-2/(\kappa-1)} u(t/\varepsilon, x/\varepsilon)$ solves the same equation with data $f_\varepsilon = \varepsilon^{-2/(\kappa-1)} f(x/\varepsilon)$, $g_\varepsilon = \varepsilon^{-2/(\kappa-1)-1} g(x/\varepsilon)$. If, say, the lifespan of $u$ were $T$, then the lifespan of $u_\varepsilon$ would be $T_\varepsilon = \varepsilon T$. On the other hand, $\|f_\varepsilon\|_{\dot{H}^\gamma} / \|f\|_{\dot{H}^\gamma} = \|g_\varepsilon\|_{\dot{H}^{\gamma-1}} / \|g\|_{\dot{H}^{\gamma-1}} = \varepsilon^{(n/2 - 2/(\kappa-1) - \gamma)}$, and so if $\gamma$ were smaller than $n/2 - 2/(\kappa - 1)$, one would have both the norm of the data and the lifespan going to zero with $\varepsilon$. If $f$ and $g$ had compact support one could thus add up suitable translates of dilates of the data, obtaining new data for which there is no local existence in a strip.

The case where $\kappa$ is the "critical power," $\kappa = (n+2)/(n-2)$, has been studied by several authors, including Grillakis [8], Pecher [23], Rauch [25], and Struwe [40]. Based on these, one would guess that, for this critical power, the scaling argument gives the right value of $\gamma$ for local existence etc., i.e. $\gamma = 1$. Pecher in fact proved that in this case there is global existence for data with small energy norms and that the scattering operator exists in a full neighborhood of the origin in $\dot{H}^1 \times L^2$ if $n \leqslant 5$. The proof of this result and many of the others concerning the critical semilinear wave equation uses space-time estimates known as Strichartz estimates. These estimates essentially say that one gains regularity because the solution to the linear Cauchy problem spreads out in all directions almost as rapidly as in the radial case, if the data is smooth enough. For powers larger than the critical one, note that when $\kappa \to \infty$ the value of $\gamma$ predicted by scaling tends to $n/2$, which is also what one expects since, for higher and higher powers of $\kappa$ one almost needs to have the $L^\infty$ norm under control to guarantee existence.

On the other hand, it was recently shown in Lindblad [21] that for $n=3$ and $\Box u = u^2$ one needs more regularity than predicted by the scaling argument, which in this special case would yield $\gamma = -1/2$. There it was shown that one has local existence if $\gamma > 0$, but the problem is ill-posed for $L^2$, i.e., $\gamma = 0$. The counterexample involves asymmetric data which forces the solution to concentrate along one light ray. In the radial case, on the other hand, it was also shown in [21] that when $n=3$, one actually has local existence for $\gamma = 0$. As a comparison, it is interesting to note that Rauch [26] for the equation $\Box u = u_t^2$ found that the lifespan for small irregular data of compact support is shorter in the nonradial case. By using the solution of one space dimension together with a domain of dependence argument, he constructs a solution that concentrates on a light ray, and he also observes that this is opposed to the radial case where the solution disperses.

In general we shall see that for high powers $\kappa$ the solution to (1.1) spreads out, while for low powers it may concentrate along a light ray. One can construct simple counterexamples that concentrate in one direction, essentially by using scaling and a Lorentz transformation. The following is a solution of (1.4)

$$u_{\varepsilon\beta}(t, x) = \frac{c_\alpha(1-\beta^2)^{\alpha/2}}{(\varepsilon - (t - \beta x_1))^\alpha}, \qquad c_\alpha = (\alpha(\alpha+1))^{\alpha/2}, \qquad \alpha = \frac{2}{\kappa - 1}, \tag{1.5}$$

which blows up when $t - \beta x_1 = \varepsilon$. Now it turns out that one can cut off initial data outside $|x| \geqslant \varepsilon$ in a suitable way so that $f_{\varepsilon\beta}(x) = u_{\varepsilon\beta}(0, x)$ and $g_{\varepsilon\beta}(x) = \partial_t u_{\varepsilon\beta}(0, x)$, when $|x| \leqslant \varepsilon$, and

$$\left( \int \big| |D_x|^\gamma f_{\varepsilon\beta}(x) \big|^2 + \big| |D_x|^{\gamma-1} g_{\varepsilon\beta}(x) \big|^2 \, dx \right)^{1/2} \leqslant C_\gamma \frac{(1-\beta)^{(n+1)/4 - \alpha/2 - \gamma}}{\varepsilon^{\gamma + \alpha - n/2}}.$$

If $\gamma < (n+1)/4 - 1/(\kappa - 1)$ we see that we can choose a sequence $\varepsilon = \varepsilon_j \to 0$ and $\beta = \beta_j \to 1$ such that the right side of this inequality tends to 0. At the same time the lifespan $T(f_{\varepsilon\beta}, g_{\varepsilon\beta}) \leqslant \varepsilon$, so we conclude that the problem is ill-posed if $\gamma < (n+1)/4 - 1/(\kappa - 1)$. Note that when $\beta \to 1$ the solution (1.5) concentrates on the hyperplane $t - \beta x_1 = \varepsilon$, this together with the cut-off of data will force the solution to concentrate on the light ray, $t = x_1$, $x_2 = \cdots = x_n = 0$, as $\varepsilon \to 0$. (See Section 6 for more details.)

The restrictions on $\gamma$ from the two different types of counterexamples, scaling and the concentration in one direction, coincide when

$\kappa=(n+3)/(n-1)$, or, equivalently, $\gamma=1/2$. And we see that the problem (1.1) is ill-posed in $\dot{H}^{\gamma}$ if

$$\gamma<\gamma(\kappa)=\begin{cases}\dfrac{n+1}{4}-\dfrac{1}{\kappa-1}, & \text{if } \kappa\leqslant\dfrac{n+3}{n-1},\\ \dfrac{n}{2}-\dfrac{2}{\kappa-1}, & \text{if } \kappa\geqslant\dfrac{n+3}{n-1},\end{cases}\tag{1.6}$$

or, equivalently,

$$\kappa>\kappa(\gamma)=\begin{cases}1+\dfrac{4}{(n+1)-4\gamma}, & \text{if } \gamma\leqslant 1/2,\\ 1+\dfrac{4}{n-2\gamma}, & \text{if } \gamma\geqslant 1/2.\end{cases}\tag{1.6$'$}$$

The endpoints $\kappa=(n+3)/(n-1)$ and $\gamma=1/2$ have a simple geometrical interpretation. The equation $\Box u=c\,|u|^{(n+3)/(n-1)}$, $c\neq 0$, is invariant under conformal transformations and so is the $\dot{H}^{1/2}$ norm of a solution to corresponding the linear equation. Therefore, we shall call the lower range "subconformal" and the upper range "superconformal." As it turns out, we have local existence if $\gamma=\gamma(\kappa)$, and $\kappa>\kappa_0$, if $\gamma(\kappa)$ is given by (1.6), and $\kappa_0=(n+1)^2/((n-1)^2+4)$, if $n\geqslant 3$, and $\kappa_0=3$ if $n=2$. If we work with the inhomogeneous norms instead we can of course have $\gamma\geqslant\gamma(\kappa)$ as well and then we only need the condition (1.2) to hold when $|u|\geqslant 1$. In higher dimensions there is a third range when $\kappa<\kappa_0$, where we have local existence if $\gamma=\gamma(\kappa)$, with the formula for $\gamma(\kappa)$ in this case being more complicated and less favorable than the one given by (1.6). When $\kappa>\kappa_0$ these local existence results were also obtained independently by Kapitanski [16]. Our results in higher dimensions for $\kappa<\kappa_0$ are stronger than his; however, for this range, we do not know whether or not they are optimal.

As we pointed out before, the improvement in the existence results for higher powers $\kappa$ (or larger $\gamma$) has to do with the fact that if, one assumes more regularity, the solution to the linear equation spreads out more like in the radial case. In a similar vein, in the radial case we have better existence results, namely, if $\gamma>\gamma_r=1/(2n)$, then, under the assumption of radial symmetry, there is local existence for (1.1) if $\kappa>\kappa_r=(n^2+4n-1)/(n^2-1)$ is the power predicted by the scaling argument, i.e., the one satisfying $\gamma=n/2-2/(\kappa-1)$.

In the superconformal range we also have global existence if the $\dot{H}^{\gamma(\kappa)}\times\dot{H}^{\gamma(\kappa)-1}$ norm of the data is smaller than a number $\varepsilon_0>0$, depending only on $n$, $\kappa$ and the constants in (1.2). There is also small-amplitude scattering and asymptotic completeness in these spaces, providing the natural extension of Pecher's results [23] concerning $\gamma=1$ and $\kappa$ the critical power.

In the radial case, we get small-amplitude global existence and scattering for the larger range where $\kappa > \kappa_r$, with $\kappa_r$ as above.

In order to obtain uniqueness one needs to make an additional assumption, which comes up naturally in the existence proof, namely,

$$\int_0^T \int_{\mathbb{R}^n} |u(t, x)|^{(\kappa-1)(n+1)/2} \, dx \, dt < \infty. \tag{1.7}$$

When $n=3$ this turns out to be sufficient for any $\gamma > 0$ (see (1.16)–(1.17) below). So, in this case, when it comes to uniqueness, there is no difference between the subconformal and superconformal cases. But if $n \geqslant 4$ and $\gamma$ is small we need to make stronger assumptions involving mixed-norms to ensure uniqueness. We do however think that (1.7) is the natural assumption since it is invariant under the scaling $a^{2/(\kappa-1)}u(at, ax)$ associated with the equation $\Box u = |u|^\kappa$. Also, the solution (1.5) just fails to satisfy (1.7), if we instead integrate only over the backward light cone $|x| \leqslant \varepsilon - t$, $0 \leqslant t \leqslant \varepsilon$. On the other hand, the solutions we construct all satisfy these mixed-norm estimates. The difficulties with uniqueness for small $\gamma$ and $n \geqslant 4$ also arise for the linear equation $\Box u = Vu$, with potential $V \in L^{(n+1)/2}([0, T] \times \mathbb{R}^n)$. In some sense it seems to be an artificial problem since the solution operator $u(0, \cdot) \to u(t, \cdot)$ defined on $C_0^\infty$ has a unique continuous extension to $\dot{H}^\gamma$.

Let us now sketch the proof of existence in the special case of the conformal wave equation $\Box u = cu^3$ in $(1+3)$ dimensions, since this is the easiest to handle and it also provides us with a model the other cases. Here, we can directly get an a priori estimate for the space time $L^4$ norm of the solution using the original Strichartz estimate [38], [39] for a solution of the linear inhomogeneous wave equation

$$\Box u = F, \qquad u(0, x) = f(x), \qquad \partial_t u(0, x) = g(x), \tag{1.8}$$

which says that, when $n \geqslant 2$,

$$\begin{aligned} &\|u\|_{L^{2(n+1)/(n-1)}(S_T)} + \|u(T, \cdot)\|_\gamma \\ &\quad \leqslant C(\|F\|_{L^{2(n+1)/(n+3)}(S_T)} + \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}), \qquad \gamma = 1/2. \end{aligned} \tag{1.9}$$

Here $S_T = [0, T] \times \mathbb{R}^n$ and

$$\|u(T, \cdot)\|_\gamma^2 = \|u(T, \cdot)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(T, \cdot)\|^2_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}.$$

To use (1.9) and solve $\Box u = cu^3$, when $n=3$ with data $(f, g) \in \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3)$, we use a simple iteration argument, setting first $u_{-1} \equiv 0$, and then defining $u_m$, $m = 0, 1, 2, \ldots$, by

$$\begin{cases} \Box u_m = cu_{m-1}^3 \\ u_m(0, x) = f(x), \qquad \partial_t u_m(0, x) = g(x). \end{cases}$$

Then, we need to show that there is a $0 < T_* \leqslant \infty$ and a function $u$ as in (3) so that

$$u_m \to u, \qquad \text{and} \qquad u_m^3 \to u^3 \qquad \text{in } \mathscr{D}'(S_{T_*}), \tag{1.10}$$

This of course implies that $u$ is a weak solution of the Cauchy problem.

The main step in proving (1.10) is to use the estimate (1.9) to see that the nonlinear map sending $u_m$ to $u_{m+1}$ is a contraction in $L^4(S_T)$ if either $T$ or the size of the data is small enough. To see this, we first notice that (1.9) and Hölder's inequality imply that if $j$, $m \geqslant -1$,

$$\begin{aligned}\|u_{m+1} - u_{j+1}\|_{L^4(S_T)} &\leqslant C\,\|u_m^3 - u_j^3\|_{L^{4/3}(S_T)} \\ &\leqslant C\,\|u_m - u_j\|_{L^4(S_T)}\,(\|u_m\|^2_{L^4(S_T)} + \|u_j\|^2_{L^4(S_T)}).\end{aligned}$$

Taking $j = -1$ yields

$$\|u_{m+1}\|_{L^4(S_T)} \leqslant \|u_{m+1} - u_0\|_{L^4(S_T)} + \|u_0\|_{L^4(S_T)} \leqslant C\,\|u_m\|^3_{L^4(S_T)} + \|u_0\|_{L^4(S_T)}. \tag{1.11}$$

The estimate (1.9) also implies that

$$\|u_0\|_{L^4(\mathbb{R}_+^{1+3})} \leqslant C(\|f\|_{\dot H^{1/2}(\mathbb{R}^3)} + \|g\|_{\dot H^{-1/2}(\mathbb{R}^3)}).$$

Therefore if the right side is small or if we take $T$ to be small enough we can assume that

$$\|u_0\|_{L^4(S_T)} \leqslant \varepsilon_0,$$

with $\varepsilon_0 > 0$ as small as we like. Moreover if this number is small enough, (1.11) and induction imply that

$$\|u_{m+1}\|_{L^4(S_T)} \leqslant \tfrac{1}{2}\,\|u_m\|_{L^4(S_T)} + \|u_0\|_{L^4(S_T)},$$

yielding $\|u_m\|_{L^4(S_T)} \leqslant 2\varepsilon_0$. On account of this we get the first part of (1.10) since if $2C\varepsilon_0^2 < 1/2$, and if we take $j = m-1$ above we find that

$$\|u_{m+1} - u_m\|_{L^4(S_T)} \leqslant \tfrac{1}{2}\,\|u_m - u_{m-1}\|_{L^4(S_T)},$$

which of course implies that $u_m$ converges to a limit $u$ in $L^4$ and hence in $\mathscr{D}'$. The fact that $u_m^3$ converges to $u^3$ follows from this and

$$\|u^3 - u_m^3\|_{L^{4/3}(S_T)} \leqslant C\,\|u - u_m\|_{L^4(S_T)} \cdot (\|u\|^2_{L^4(S_T)} + \|u_m\|^2_{L^4(S_T)}).$$

In fact, since we have just seen that the $L^4$ norms of $u$ and the $u_m$ are bounded by a fixed constant, this inequality together with the convergence of $u_m$ to $u$ in $L^4$ implies that $u_m^3$ converges to $u^3$ in $L^{4/3}$ and hence in $\mathscr{D}'$.

Finally, the remaining fact that the $\dot{H}^{1/2}$ norms of $u(t, \cdot)$ are uniformly bounded for $0 < t < T$, follows from another application of (1.9).

One can prove asymptotic completeness and the existence of the scattering operator for small-amplitude $\dot{H}^{1/2}$ solutions in a similar manner. For instance, if $u$ as above is a global solution of $\Box u = u^3$, then, since $u \in L^4(\mathbb{R}^{1+3}_+)$, one can use (1.9) to see that if $T_j \to +\infty$ and $u_j$ solves the free wave equation with data $(u(T_j, \cdot), \partial_t u(T_j, \cdot))$, then $(u_j(0, \cdot), \partial_t u_j(0, \cdot))$ is a Cauchy sequence of initial data in $\dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3)$. If $u_+$ is the solution to the free wave equation with the limiting data, it is not hard to use (1.9) to see that $\lim_{T \to +\infty} \|u(T, \cdot) - u_+(T, \cdot)\|_{\dot{\gamma}} = 0$, $\gamma = 1/2$. The existence of the scattering operator near the origin follows more directly from the proof of global existence for small data.

The proof that there is a unique solution with the above properties follows the same lines. For instance, if $u$ and $\tilde{u} \in L^4(S_{T_*})$ both solve the conformally invariant equation in $S_{T_*} \subset \mathbb{R}^{1+3}$ with the same data in $\dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3)$, one argues as above to see that $\|u - \tilde{u}\|_{L^4(S_T)} \leqslant \frac{1}{2} \|u - \tilde{u}\|_{L^4(S_T)}$, provided that $T > 0$ is small enough. This of course implies that $u = \tilde{u}$ in $S_T$, leading to the uniqueness.

The proof of our existence results for the other cases follows similar lines, except that we need to use an estimate which generalizes the original Strichartz estimate (1.9). Specifically, if now $u$ solves the inhomogeneous Cauchy problem (1.8), then there is a constant $C_q$ depending only on $q$ so that

$$\|u\|_{L^s_t L^q_x(S_T)} + \|u(T, \cdot)\|_{\dot{\gamma}} \leqslant C_q(\|F\|_{L^r_t L^p_x(S_T)} + \|f\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}), \tag{1.9'}$$

if we have the gap condition

$$n\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{r} - \frac{1}{s} = 2, \tag{1.12}$$

and

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}, \qquad s = \frac{4q}{(n-1)(q-2)}, \qquad \gamma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right),$$

$$\text{when} \begin{cases} \left|\frac{1}{2} - \gamma\right| < \frac{1}{n-1}, & n \geqslant 3, \\ \left|\frac{1}{2} - \gamma\right| < \frac{1}{4}, & n = 2. \end{cases} \tag{1.13}$$

Also, if $n \geqslant 2$, $2(n+1)/(n-1) \leqslant q < \infty$ and $\gamma = n/2 - (n+1)/q \geqslant 1/2$

$$\begin{aligned} &\|u\|_{L^q(S_T)} + \| \, |D_x|^{\gamma - 1/2} u \, \|_{L^{2(n+1)/(n-1)}(S_T)} + \|u(T, \cdot)\|_{\dot{\gamma}} \\ &\quad \leqslant C_q(\| \, |D_x|^{\gamma - 1/2} F \|_{L^{2(n+1)/(n+3)}(S_T)} + \|f\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}). \end{aligned} \tag{1.14}$$

Here $\|u\|^s_{L^s_t L^q_x(S_T)} = \int_0^T \|u(t, \cdot)\|^s_{L^q_x(\mathbb{R}^n)}\, dt$. These inequalities will be proved in Section 3.

The special case of (1.9′) where $\gamma = \frac{1}{2}$ and $q = 2(n+1)/(n-1)$ is just the Strichartz estimate (1.9). The condition (1.12) is necessary because of scaling considerations. Notice that the relations between the exponents in (1.13) are such that if $F = Vu$ then

$$\|F\|_{L^r_t L^p_x(S_T)} \leqslant \|V\|_{L^{(n+1)/2}(S_T)} \|u\|_{L^s_t L^q_x(S_T)}, \tag{1.15}$$

which is how the condition (1.7) comes up. Moreover, in the upper range $q \geqslant 2(n+1)/(n-1)$, the $L^r_t L^p_x \to L^s_t L^q_x$ estimate for the solution to the inhomogeneous equation with zero data follows by duality from the same estimate in the lower range $q \leqslant 2(n+1)/(n-1)$.

Using these estimates it is not hard to adapt the proof of the existence result sketched above for the special case where $\kappa = 3$ and $n = 3$. For example, if $n = 3$, one can use (1.9′) and (1.15) to conclude that, in the subconformal range (when $0 < \gamma < 1/2$), the map sending $u_m$ to $u_{m+1}$ is a contraction in the space $L^s_t L^q_x$ if $\|u_0\|_{L^{2(\kappa-1)}(S_T)}$ is sufficiently small. By (1.13), $q = 2(\kappa - 1)$ if $\gamma = 1 - 1/(\kappa - 1) = \gamma(\kappa)$ where $\gamma(\kappa)$ is given by the first case in (1.6). Similarly, in the superconformal range, $\gamma > 1/2$, one uses the estimate (1.14) to conclude that we have a contraction in the space defined by the norms in the left side of (1.14), if the above condition is satisfied, and this yields $\gamma = 3/2 - 2/(\kappa - 1) = \gamma(\kappa)$. In higher dimensions $n \geqslant 4$ one also needs another estimate similar to (1.9′) when $0 \leqslant \gamma < 1/2 - 1/(n-1)$, whose statement we postpone until Section 3.

The same argument used to prove uniqueness for solutions satisfying (1.7) when $\kappa = 3$ now gives a uniqueness result for the linear Cauchy problem with a potential. This in turn will directly give us uniqueness for solutions of the nonlinear problem satisfying (1.7). Suppose that $V \in L^2([0, T] \times \mathbb{R}^3)$ and that $(f, g) \in \dot{H}^\gamma(\mathbb{R}^3) \times \dot{H}^{\gamma-1}(\mathbb{R}^3)$, with $0 < \gamma < 1$. Then, using (1.9′) and (1.15) one can prove that the equation

$$\Box u = Vu, \qquad u(0, x) = f(x), \qquad \partial_t u(0, x) = g(x). \tag{1.16}$$

has a unique solution $u$ for which $(u, \partial_t u) \in C([0, T];\ \dot{H}^\gamma(\mathbb{R}^3) \times \dot{H}^{\gamma-1}(\mathbb{R}^3))$. Moreover, if $0 < t < T$, there is a universal constant $K_\gamma$ so that

$$\|u(t, \cdot)\|_{\dot{\gamma}} \leqslant 2 \exp\left(K_\gamma \int_0^t \int_{\mathbb{R}^3} |V(s, x)|^2\, dx\, ds\right) \cdot \|u(0, \cdot)\|_{\dot{\gamma}}. \tag{1.17}$$

These results greatly improve the regularity assumptions for the data of other uniqueness theorems for singular hyperbolic equations, for instance in Sogge [32], and this is of course important for nonlinear applications. The natural assumption for the potential is the same as in [32] and this also turns out to be important for applications. For instance in the $(3+1)$-dimensional case, if $0<\gamma(\kappa)<1$, one immediately gets the uniqueness for solutions satisfying (1.7), discussed above: If $u$ and $\tilde{u}$ both solve both solve (1.1) with the same data, then $u-\tilde{u}$ must vanish identically since it has zero data and $\Box(u-\tilde{u})=Vu$, where $V=(F_\kappa(u)-F_\kappa(\tilde{u}))/(u-\tilde{u})$ is in $L^2(S_{T_*})$, due to the fact that $u, \tilde{u} \in L^{2(\kappa-1)}(S_{T_*})$. The inequality (1.17) also shows that regularity propagates as long as (1.7) is true, so if the solution blows up then the integral in (1.7) has to become infinite. Also the integral (1.7) over a backward light cone has to become infinite, see Section 5.

The proofs of the estimates (1.9′), (1.14) and similar estimates for higher dimensions are related to the proofs of smoothing estimates from harmonic analysis. Specifically, using the form of the fundamental solution of $\Box$ one sees that, in order to prove the estimates (1.9′)–(1.14), it suffices to make appropriate mixed-norm estimates for operators of the form

$$(W^\alpha F)(t, x)=\iint_{\mathbb{R}^{1+n}} e^{ix\cdot\xi+i(t-s)|\xi|}\hat{F}(s, \xi)\,\frac{d\xi}{|\xi|^\alpha}\,ds, \qquad \alpha<n,$$

or for related operators sending functions of $n$-variables to functions of $(n+1)$-variables. The proofs only use the Hardy-Littlewood theorem for fractional integrals, the M. Riesz interpolation theorem and pointwise estimates for the dyadic parts of the kernels:

$$K_j^\alpha(t, x)=\int_{\mathbb{R}^n} e^{ix\cdot\xi+it|\xi|}\beta(|\xi|/2^j)\,\frac{d\xi}{|\xi|^\alpha},$$

if $\beta\in C_0^\infty(\mathbb{R}_+^n)$. The pointwise estimates, which are related to Huygen's principle, are the following:

$$|K_j^\alpha(t, x)|\leqslant C_N\lambda^{(n+1)/2-\alpha}(|t|+\lambda^{-1})^{-(n-1)/2}(1+\lambda\,\big|\,|t|-|x|\,\big|)^{-N},$$
$$N=1, 2, ..., \lambda=2^j.$$

These follow easily from stationary phase. The large negative power of $|t|$ in higher dimensions, which is related to the fact that the fundamental solution becomes more and more singular as $n$ increases, accounts for why the favorable range in the estimate (1.9′) gets smaller and smaller as $n$ grows.

The paper is organized as follows. In the next section we give precise statements of our main results and in Section 3 we state and prove the inequalities needed for their proof. In Section 4 we prove the main existence results, while sections 5 and 6 are devoted to uniqueness and sharpness, respectively. In Section 7 we show that there is small-amplitude scattering and asymptotic completeness for superconformal equations, while in Section 8 we show that all of our results improve somewhat if one only considers radially symmetric data.

It is a pleasure to thank the many people who have helped us with this work. In particular we are grateful to J. Ginibre, S. Klainerman, K. Okikiolu, M. Machedon, J. Rauch, A. Seeger, I. Segal, A. Soffer, and M. Zworski for helpful suggestions and conversations.

## 2. Main Results

Our main results concern sharp local existence for (1.1) in a strip $S_T = [0, T] \times \mathbb{R}^n$ and global existence and scattering for small data in the superconformal case.

THEOREM 2.1. *Let $n \geqslant 2$ and set*

$$\kappa_0 = \frac{(n+1)^2}{(n-1)^2+4}, \quad \text{if} \quad n \geqslant 3, \quad \text{and} \quad \kappa_0 = 3 \quad \text{for} \quad n = 2.$$

*Assume that $F_\kappa$ satisfies (1.2) for $\kappa_0 < \kappa < \infty$ if $n = 2$ or 3, or $\kappa_0 < \kappa \leqslant (n+1)/(n-3)$ for $n \geqslant 4$. If $n \geqslant 4$ and $\kappa > (n+1)/(n-3)$ we may also take $F_\kappa = \pm u^\kappa$, provided that $\kappa$ is an integer. Suppose that the initial data satisfy $f \in \dot{H}^\gamma(\mathbb{R}^n)$, $g \in \dot{H}^{\gamma-1}(\mathbb{R}^n)$, with $\gamma$ given by*

$$\gamma < \gamma(\kappa) = \begin{cases} \dfrac{n+1}{4} - \dfrac{1}{\kappa-1}, & \kappa_0 < \kappa \leqslant \dfrac{n+3}{n-1}, \\[2mm] \dfrac{n}{2} - \dfrac{2}{\kappa-1}, & \text{if} \quad \kappa \geqslant \dfrac{n+3}{n-1}. \end{cases} \tag{2.1}$$

*Then there is a $T_* > 0$ and a unique (weak) solution $u$ to (1.1) verifying*

$$(u, \partial_t u) \in C([0, T_*]; \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)) \quad \text{and} \quad u \in L_t^s L_x^q([0, T_*] \times \mathbb{R}^n), \tag{2.2}$$

*with $q = (\kappa-1)(n+1)/2$ and $s = q$ for $\kappa \geqslant (n+3)/(n-1)$ and $s = 4q/(n-1)(q-2)$ for $\kappa_0 < \kappa \leqslant (n+3)/(n-1)$. For a given $\kappa < (n+3)/(n-1)$, $T_*$ depends only on the size of the norm of the initial data, while for $\kappa \geqslant (n+3)/(n-1)$ this is not the case.*

THEOREM 2.2. *Let $F_\kappa$, $\kappa \geqslant (n+3)/(n-1)$, and $\gamma = \gamma(\kappa)$ be as in Theorem 2.1. Then, if the norm of the data is small, i.e.,*

$$\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} < \varepsilon, \tag{2.3}$$

*(with $\varepsilon > 0$ depending only on $n$, $\kappa$, and the constants in* (1.2)), *there is a unique global (weak) solution to* (1.1) *verifying*

$$(u, \partial_t u) \in C_b(\mathbb{R}; \dot{H}^{\gamma}(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)) \quad \text{and} \quad u \in L^{(\kappa-1)(n+1)/2}(\mathbb{R}^{1+n}). \tag{2.4}$$

*Moreover, for given data $(f, g)$ as above, there is data $(f_+, g_+) \in \dot{H}^{\gamma} \times \dot{H}^{\gamma-1}$ so that the (weak) solution to the free wave equation with this data,*

$$\begin{cases} \Box u_+ = 0, \\ u_+(0, x) = f_+(x), \qquad \partial_t u_+(0, x) = g_+(x), \end{cases}$$

*satisfies*

$$\lim_{T \to +\infty} \|u(T, \cdot) - u_+(T, \cdot)\|_{\dot{\gamma}} = 0.$$

*Conversely, if $(f_-, g_-) \in \dot{H}^{\gamma} \times \dot{H}^{\gamma-1}$ has small norm and if $u_-$ is the solution to the free wave equation with this data, then there is a solution $u$ to* (1.1) *satisfying* (2.4) *and*

$$\lim_{T \to -\infty} \|u(T, \cdot) - u_-(T, \cdot)\|_{\dot{\gamma}} = 0.$$

*Thus, the scattering operator $S : (f_-, g_-) \to (f_+, g_+)$ exists in a neighborhood of the origin in $\dot{H}^{\gamma} \times \dot{H}^{\gamma-1}$.*

An equivalent way of stating the local existence part of Theorem 2.1 is that for data $f \in \dot{H}^{\gamma}(\mathbb{R}^n)$, $g \in \dot{H}^{\gamma-1}(\mathbb{R}^n)$ there is local existence for (1.1), provided that

$$\kappa < \kappa(\gamma) = \begin{cases} 1 + \dfrac{4}{(n+1) - 4\gamma}, & \text{if} \quad \gamma_0 < \gamma \leqslant 1/2, \\ 1 + \dfrac{4}{n - 2\gamma}, & \text{if} \quad \gamma \geqslant 1/2, \end{cases} \tag{2.1$'$}$$

with $\gamma_0 = (n-3)/2(n-1)$, $n \geqslant 3$ and $\gamma_0 = 1/4$ for $n = 2$. As we shall see, the different behavior for $\gamma$ smaller and bigger than $1/2$ is related to the Strichartz-1/2 local smoothing estimate for the wave equation. See [39], [31, p. 215]. It is interesting to note that, since $1/2 = \gamma(\kappa) \Leftrightarrow \kappa = (n+3)/(n-1)$, the change of behavior takes place at the conformally invariant nonlinearity (e.g., $F_\kappa(u) = |u|^{(n+3)/(n-1)}$).

For the local existence results, we can also take $\gamma > \gamma(\kappa)$ in (2.1) if we use assume that the data belong to the inhomogeneous Sobolev spaces $H^{\gamma}$ and $H^{\gamma-1}$, respectively. Under this assumption we can also of course assume that $\kappa$ is smaller than the number given in (2.1′). Moreover, if we assume that the data belong to the inhomogeneous Sobolev spaces we need only assume that (1.2) holds when $|u| \geqslant 1$. On the other hand, assuming that the data belong to the homogeneous Sobolev spaces and that (1.2) holds for small $u$ as well is necessary for the global existence results. Finally, if $n \geqslant 4$ and $\kappa > (n+1)/(n-3)$, we have to assume that $\kappa$ is an integer and that $F_\kappa$ is a pure power for technical reasons based on the fact that our proof requires a certain amount of regularity of $F_\kappa$ if $\kappa$ is larger than $(n+3)/(n-1)$. On the other hand, because of the nonlinear relationship between $\kappa$ and $\gamma(\kappa)$, if $\kappa$ were larger than a constant depending on the dimensions, we could drop the assumption about $F_\kappa$ being a pure power, with $\kappa$ an integer.

The local existence results generalize those in Lindblad [21]. There it was shown that for $n=3$ and $\square u = F_\kappa(u)$ with $\kappa = 2$ there is local existence if $\gamma > 0$. On the other hand, it was shown that this problem is not well-posed for $L^2$, i.e., $\gamma = 0$, so it is interesting that for $n=3$ and $\kappa > 0$ one can obtain sharp endpoint results. It is also worth noting that in the range $\kappa \geqslant (n+3)/(n-1)$ the existence is just what is given by the trivial scaling argument, whereas in the lower range $\kappa < (n+3)/(n-1)$ one needs more regularity than predicted by the scaling argument. Some of these results were also obtained independently by Kapitanski [16] using a different proof. Also influenced by Lindblad's work, he obtained the local existence results in Theorem 2.1 when $n \geqslant 3$ and $\kappa_0 < \kappa \leqslant (n+2)/(n-2)$. The results for $\kappa > (n+2)/(n-2)$ and the global existence and scattering results, i.e., Theorem 2.2, are new.

Our results also improve some in Beals and Bezard [1]. In this paper, they showed that for $n \geqslant 5$ there is local existence for $F_\kappa(u) = u^2$ provided that $\gamma = (n-3)/2$, while our results show that this is the case if $\gamma = 1/4$, when $n = 4$, or $(n-4)/2$ if $n \geqslant 5$.

The scattering results in Theorem 2.2 are the natural extension of those in Pecher [23] who handled the case where $\gamma(\kappa) = 1$, and $n \leqslant 5$. See also Strauss [36], [37].

In higher dimensions there is a third range of $\kappa$. Here the relationship between $\kappa$ and $\gamma$ is less favorable than the above one corresponding to $\kappa_0 < \kappa \leqslant (n+3)/(n-1)$. As we shall see, this is related to the fact that the $\dot{H}^\gamma(\mathbb{R}^n)$ estimates for the (linear) wave equation are less favorable for $\gamma$ smaller than $\gamma_0 = (n-3)/2(n-1)$, compared to $\gamma > \gamma_0$ (cf. Corollary 3.4).

THEOREM 2.3. *Let $n \geqslant 4$ and suppose that $(n+3)/n \leqslant \kappa < \kappa_0 = (n+1)^2/((n-1)^2+4)$. Suppose further that $f \in \dot{H}^\gamma(\mathbb{R}^n)$, $g \in \dot{H}^{\gamma-1}(\mathbb{R}^n)$, with $\gamma$ satisfying*

$$\gamma = \gamma(\kappa) = \frac{n+1}{4} - \frac{(n+1)(n+5)}{4} \cdot \frac{1}{2n\kappa - (n+1)}. \tag{2.5}$$

*Then there is a $T_* > 0$, depending only on the size of the initial data, and a unique (weak) solution $u$ to (1.1) verifying*

$$(u, \partial_t u) \in C([0, T_*]; \dot{H}^{\gamma}(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)) \quad \text{and} \quad u \in L_t^s L_x^q([0, T_*] \times \mathbb{R}^n), \tag{2.6}$$

*with $q = (4n\kappa - 2(n+1))/(n+5)$ and $s = 4q/((n-1)(q-2))$.*

As before, we can restate things in terms of $\gamma$. Specifically, if $f \in \dot{H}^{\gamma}(\mathbb{R}^n)$ and $g \in \dot{H}^{\gamma-1}(\mathbb{R}^n)$, there is local existence for (1.1) provided that

$$\kappa = \frac{n+1}{n}\left(1 + \frac{2(1+\gamma)}{(n+1) - 4\gamma}\right), \qquad \text{if} \quad 0 \leqslant \gamma < \frac{n-3}{2(n-1)}. \tag{2.5'}$$

Also, if $\gamma > \gamma(\kappa)$, we have local existence and uniqueness if we assume that the data belong to the inhomogeneous Sobolev spaces $H^{\gamma}$ and $H^{\gamma-1}$, respectively. Finally, for the border case where $\kappa = \kappa_0$, the proof of Theorem 2.3 also shows that if $\gamma > \gamma(\kappa_0) = (n-3)/2(n-1)$, there is local existence and uniqueness for $f \in H^{\gamma}(\mathbb{R}^n)$ and $g \in H^{\gamma-1}(\mathbb{R}^n)$.

Theorem 2.3 is stronger than corresponding results in Kapitanski [16]. In particular, for $L^2$ data, i.e. $\gamma = 0$, he just shows that there is local existence if $\kappa < (n+1)/(n-1) < (n+3)/n$. We do not know, however, if $(n+3)/n$ is the sharp power for local existence when $\gamma = 0$.

## 3. Inequalities

The purpose of this section is to collect the inequalities which we shall need to prove our main local existence and uniqueness results. Specifically, we shall be interested in estimates for the inhomogeneous Cauchy problem

$$\begin{cases} \Box u(t, x) = F(t, x) \\ u(0, x) = f(x), \qquad \partial_t u(0, x) = g(x). \end{cases} \tag{3.1}$$

Recall that $u$ can be written as

$$u(t, x) = v(t, x) + w(t, x),$$

where $v$ solves the homogeneous Cauchy problem,

$$\begin{cases} \Box v(t, x) = 0 \\ v(0, x) = f(x), \qquad \partial_t v(0, x) = g(x), \end{cases} \tag{3.2}$$

and $w$ solves the inhomogeneous wave equation with zero data,

$$\begin{cases} \Box w(t, x) = F(t, x) \\ w(0, x) = \partial_t w(0, x) = 0. \end{cases} \tag{3.3}$$

Thus our estimates for (3.1) really split into two types: one for the homogeneous problem (3.2) and one for the inhomogeneous problem (3.3). As before, $\dot{H}^\gamma(\mathbb{R}^n)$ denotes the homogeneous Sobolev space with norm

$$\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \| |D_x|^\gamma f\|_{L^2(\mathbb{R}^n)},$$

where

$$|D_x| = \sqrt{-\Delta_x}.$$

We start out with the homogeneous estimates we shall need. They are straightforward and hold in spatial dimensions $\geqslant 2$.

THEOREM 3.1. *Let $n \geqslant 2$ and let $v$ solve the Cauchy problem* (3.2). *Then*

$$\begin{gathered} \|v\|_{L^q(\mathbb{R}^{1+n})} \leqslant C(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}), \\ \frac{2(n+1)}{n-1} \leqslant q < \infty, \qquad \gamma = \frac{n}{2} - \frac{n+1}{q}. \end{gathered} \tag{3.4}$$

*On the other hand, if $2 \leqslant q < \infty$ when $n = 2, 3$ or $2 \leqslant q < 2(n-1)/(n-3)$ for $n \geqslant 4$,*

$$\begin{gathered} \|v\|_{L_t^s L_x^q(\mathbb{R}^{1+n})} \leqslant C(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}), \\ s = \frac{4q}{(n-1)(q-2)}, \qquad \gamma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \end{gathered} \tag{3.5}$$

*while, if $n \geqslant 4$ and $2(n-1)/(n-3) < q \leqslant 2n/(n-3)$,*

$$\|v\|_{L_t^2 L_x^q(\mathbb{R}^{1+n})} \leqslant C(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}), \qquad \gamma = \frac{n-1}{2} - \frac{n}{q}. \tag{3.6}$$

*Finally,*

$$\|v(T, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t v(T, \cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \leqslant 2(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}). \tag{3.7}$$

In addition to these we shall need the following estimates for the inhomogeneous problem.

THEOREM 3.2. *Let $n \geqslant 2$ and suppose that $w$ solves (3.3) in $S_T = [0, T] \times \mathbb{R}^n$. Then for $T > 0$,*

$$\|w\|_{L^q(S_T)} \leqslant C_q \, \| |D_x|^{(n-1)/2-(n+1)/q} F\|_{L^{2(n+1)/(n+3)}(S_T)}, \qquad \frac{2(n+1)}{n-1} \leqslant q < \infty, \tag{3.8}$$

*where, for a given $n$, $C_q$ remains bounded if $q$ is smaller than a fixed exponent. Also,*

$$\|w\|_{L_t^s L_x^q(S_T)} \leqslant C_p \, \|F\|_{L_t^r L_x^p(S_T)}, \tag{3.9}$$

*provided that $2(n-1)/(n+1) < p < 2(n+1)(n-1)/((n+1)^2 - 8)$ if $n \geqslant 3$, $1 < p < 3/2$ if $n = 2$, and*

$$q = \frac{(n+1)p}{(n+1) - 2p}, \qquad r = \frac{4p}{(n+3)p - 2(n-1)},$$

*and* (3.10)

$$s = \frac{4(n+1)p}{(n+5)(n-1)p - 2(n+1)(n-1)}.$$

*Finally,*

$$\|w(T, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t w(T, \cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \leqslant C \, \| |D_x|^{\gamma - 1/2} F\|_{L^{2(n+1)/(n+3)}(S_T)}, \tag{3.11}$$

*and*

$$\|w(T, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t w(T, \cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \leqslant C_p \, \|F\|_{L_t^r L_x^p(S_T)}, \qquad \max\left\{\frac{2(n-1)}{n+1}, 1\right\} < p \leqslant 2, \tag{3.12}$$

*if*

$$\gamma = 1 - \frac{n+1}{4}\left(\frac{1}{p} - \frac{1}{p'}\right) = \frac{n+5}{4} - \frac{n+1}{2p}, \qquad \text{and} \qquad r = \frac{4p}{(n+3)p - 2(n-1)}. \tag{3.13}$$

Notice that the range of exponents $(s, q)$ in (3.5) is dual to the one corresponding to the exponents $(r, p)$ in (3.12). As we shall see, the proofs of the two inequalities are related. Also, the exponents $p$ in (3.9) are "symmetric" around $p = 2(n+1)/(n+3)$. Specifically $p$ is as in (3.9) if and only if $|1/p - (n+3)/2(n+1)| < 2/(n+1)(n-1)$, for $n \geqslant 3$ and $< 1/6$ if $n = 2$. Since the exponents $(s, q)$ in (3.9) are dual to the exponents $(r, p)$ there, we similarly have that $|1/q-(n-1)/2(n+1)|$ satisfies the same bounds. With this in mind, it is not surprising that the proof of (3.9) should be related our proof of (3.8).

One can express the relationship between the exponents in the mixed-norm inequalities in a more palatable manner. Specifically, the exponents in (3.10) are determined by the relations

$$s=\frac{4q}{(n-1)(q-2)}, \qquad n\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{r}-\frac{1}{s}=2, \qquad \frac{1}{p}-\frac{1}{q}=\frac{1}{r}-\frac{1}{s}. \tag{3.10'}$$

The second condition is of course necessary by scaling considerations. Using (3.10′) one also sees that the exponents in equality (3.9) satisfy the natural gap condition

$$1/p-1/q=1/r-1/s=2/(n+1).$$

Some of the estimates for the homogeneous equation were known before. Inequality (3.4) is equivalent to the Strichartz restriction theorem for the Fourier transform [39]. Inequality (3.5) is due to Pecher [23]. We shall give a slightly different proof of these inequalities that will be useful for proving the inhomogeneous inequalities.

Inequality (3.9) generalizes inequalities in Lindblad [21], and Pecher [23]. A counterexample in [21] shows that (3.9) cannot hold when $n=3$ and $p=1$. (See also Klainerman and Machedon [19].) Similar considerations apply to the endpoints in other dimensions.

L. Kapitanski has kindly pointed out that some of the estimates in Theorem 3.2 follow from his earlier work. For instance (3.8) and (3.11) follow from Lemma 1.5 in [14, III]. Alternatively these follow easily from the original Strichartz estimates in [38], [39]. Also, Kapitanski informed us that when $p\leqslant 2(n+1)/(n+3)$, (3.9) follows from his [13, Corollary 7.4] and that the same result implies the related inequality (3.12). Our proof of all of these inequalities, and indeed all the ones in Theorems 3.1 and 3.2, will use ideas from proof of the $L^2$ restriction theorems of Stein, Tomas, and Strichartz (cf. [39]).

Theorems 3.1 and 3.2 contain all the estimates we need to prove Theorem 2.1 and the part of Theorem 2.3 corresponding to $\kappa>(n+1)^2/([(n-1)^2+2^2]$. To handle the lower range of $\kappa$ in the latter theorem we need one more set of inequalities for the wave equation which only occurs in higher dimensions.

THEOREM 3.3. *Let $n\geqslant 4$ and suppose that $w$ solves (3.3). Then for $T>0$,*

$$\|w\|_{L_t^s L_x^q(S_T)}\leqslant C_p\,\|F\|_{L_t^2 L_x^p(S_T)}, \qquad \frac{2n}{n+3}\leqslant p<\frac{2(n-1)}{n+1}, \tag{3.14}$$

*if*

$$q = \frac{2(n+1)p}{4n-(n+5)p}, \qquad \textit{and} \qquad s = \frac{2(n+1)p}{(n+3)(n-1)p-2n(n-1)}, \tag{3.15}$$

*or*

$$\|w\|_{L_t^2 L_x^q(S_T)} \leqslant C_p \|F\|_{L_t^r L_x^p(S_T)}, \qquad \frac{2(n+1)(n-1)}{(n+1)^2-8} < p \leqslant 2, \tag{3.16}$$

*with*

$$q = \frac{4np}{(n-7)p+2(n+1)}, \qquad r = \frac{4p}{4p-(n-1)(2-p)}. \tag{3.17}$$

*Also,*

$$\begin{gathered} \|w(T,\cdot)\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\partial_t w(T,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} \leqslant C_p \|F\|_{L_t^2 L_x^p(S_T)}, \\ \frac{2n}{n+3} \leqslant p < \frac{2(n-1)}{n+1}, \qquad \gamma = n\left(\frac{n+3}{2n} - \frac{1}{p}\right). \end{gathered} \tag{3.18}$$

As in the previous theorem, one can rewrite the relations between the exponents in (3.14). Since $r=2$ in (3.14), (3.15) can be rewritten as

$$s = \frac{4q}{(n-1)(q-2)}, \qquad n\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{2} - \frac{1}{s} = 2. \tag{3.15$'$}$$

This is consistent with (3.10′) above, except here we no longer have $1/p - 1/q = 1/r - 1/s$.

Also, like in the mixed-norm estimates in Theorem 3.3, the exponents $p$ in (3.14) and (3.16) are "symmetric" around $p = 2(n+1)/(n+3)$. Specifically, here, they satisfy $2/(n+1)(n-1) < |1/p - (n+3)/2(n+1)| < 1/(n+1)$.

If we combine the estimates in Theorems 3.1–3.3 we get the following.

COROLLARY 3.4. *Let $n \geqslant 2$ and let $u$ solve the inhomogeneous Cauchy problem (3.1). Then there is a constant $C_q$ depending only on $q$ so that*

$$\begin{aligned} &\|u\|_{L_t^s L_x^q(S_T)} + \|u(T,\cdot)\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} \\ &\quad \leqslant C_q(\|F\|_{L_t^r L_x^p(S_T)} + \|f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}), \end{aligned} \tag{3.19}$$

*if we have the gap condition*

$$n\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{r} - \frac{1}{s} = 2,$$

*and*

$$\text{(i)}\quad \frac{1}{p}-\frac{1}{q}=\frac{2}{n+1},\qquad s=\frac{4q}{(n-1)(q-2)},\qquad \gamma=\frac{n+1}{2}\left(\frac{1}{2}-\frac{1}{q}\right),$$

$$\text{when}\quad \begin{cases} \left|\frac{1}{2}-\gamma\right|<\frac{1}{n-1}, & n\geqslant 3,\\ \left|\frac{1}{2}-\gamma\right|<\frac{1}{4}, & n=2. \end{cases}$$

*If* $n\geqslant 4$ *the inequality also holds if the gap condition holds and*

$$\text{(ii)}\quad r=2,\qquad s=\frac{4q}{(n-1)(q-2)},\qquad \gamma=\frac{n+1}{2}\left(\frac{1}{2}-\frac{1}{q}\right),$$

$$0\leqslant\gamma<\frac{n-3}{2(n-1)},$$

*or*

$$\text{(iii)}\quad s=2,\qquad r=\frac{4p}{4p-(n-1)(2-p)},\qquad \gamma=\frac{n-1}{2}-\frac{n}{q},$$

$$\frac{n+1}{2(n-1)}<\gamma\leqslant 1.$$

*Also, if* $n\geqslant 2$, $2(n+1)/(n-1)\leqslant q<\infty$ *and* $\gamma=n/2-(n+1)/q$

$$\|u\|_{L^q(S_T)}+\|\,|D_x|^{\gamma-1/2}u\|_{L^{2(n+1)/(n-1)}(S_T)}+\|u(T,\cdot)\|_{\dot H^\gamma(\mathbb{R}^n)}+\|\partial_t u(T,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}$$
$$\leqslant C_q(\|f\|_{\dot H^\gamma(\mathbb{R}^n)}+\|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}+\|\,|D_x|^{\gamma-1/2}F\|_{L^{2(n+1)/(n+3)}(S_T)}). \tag{3.20}$$

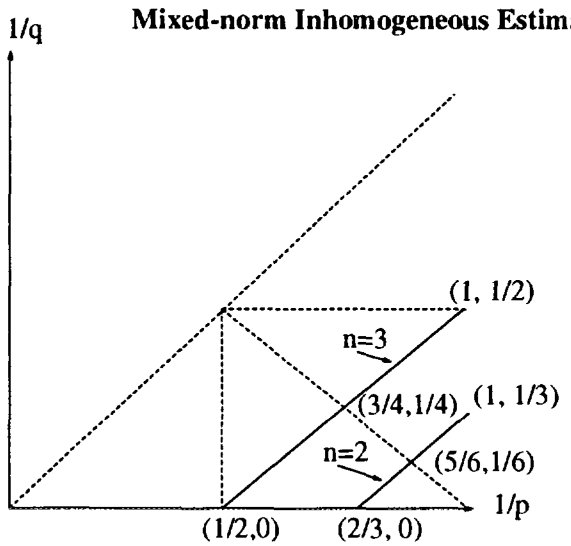
One could restate the conditions on $\gamma$ in the three cases in terms of a condition on $q$ (or $p$). In case (i) it would read:

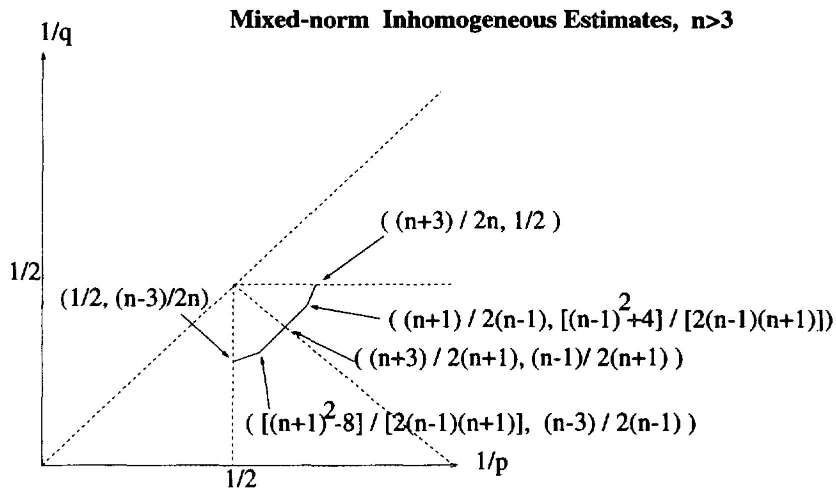
$$\begin{cases} 3<q<\infty, & n=2\\ 2<q<\infty, & n=3\\ \frac{2(n+1)(n-1)}{(n-1)^2+4}<q<\frac{2(n-1)}{n-3}, & n\geqslant 4 \end{cases} \tag{3.21}$$

while in the other two cases it would become $2\leqslant q<2(n+1)(n-1)/((n-1)^2+4)$ and $2(n-1)/(n-3)<q\leqslant 2n/(n-3)$, respectively.

The inequality in case (iii) generalizes the estimates of Pecher [23], who proved the estimate corresponding to $\gamma=1$ when $n=3$, and a slightly weaker version in higher dimensions. Such inequalities have recently been used by Grillakis [9] and Kapitanski [14] to study the critical semilinear wave equation. A related approach based on (3.20) was employed by Shatah and Struwe [30].

In the following two figures we graph $(1/p, 1/q)$, where $p$ and $q$ are pairs of exponents occurring in the above mixed-norm estimates.




Before turning to the proofs, let us isolate a simple consequence of the inequalities for the inhomogeneous equation that will be useful in the proof of the local existence and uniqueness theorems as well as other results to follow.

**Proposition** 3.5. *Let $n \geqslant 3$ and assume that $w$ is a solution of*

$$\Box w = Vu, \qquad w(0, x) = w_t(0, x) = 0, \qquad \textit{in} \quad S_T = \{(t, x) \in \mathbb{R}^{1+n} : 0 \leqslant t < T\}.$$

*Then*

$$\|w\|_{L_t^s L_x^q(S_T \cap \Lambda_{R,0})} + \|w\|_{L_t^\infty L_x^e(S_T \cap \Lambda_{R,0})} \leqslant C_q \|V\|_{L^{(n+1)/2}(S_T \cap \Lambda_{R,0})} \|u\|_{L_t^s L_x^q(S_T \cap \Lambda_{R,0})}, \tag{3.22}$$

*if $q$ is as in* (3.21),

$$s = \frac{4q}{(n-1)(q-2)}, \qquad \frac{n}{e} = \frac{n}{q} + \frac{1}{s},$$

*and $\Lambda_{R,0} = \{(t, x) \in \mathbb{R}^{1+n} : |x| < R - t,\ t \geqslant 0\}$ and $0 < R \leqslant \infty$. On the other hand, assume that $w$ is a solution of*

$$\Box w = F(u), \qquad w(0, x) = w_t(0, x) = 0,$$

*then, if $q$ is finite and $1/2 \leqslant \gamma = n/2 - (n+1)/q \leqslant 3/2$, we have*

$$\begin{aligned} &\|w\|_{L^q(S_T)} + \| |D_x|^{\gamma - 1/2} w\|_{L^{2(n+1)/(n-1)}(S_T)} \\ &\qquad \leqslant C_q \|F'(u)\|_{L^{(n+1)/2}(S_T)} \| |D_x|^{\gamma - 1/2} u\|_{L^{2(n+1)/(n-1)}(S_T)}. \end{aligned} \tag{3.23}$$

*Proof of Proposition* 3.5. The proof is just an application of the estimates in Corollary 3.4. Let us first prove (3.22) when $R = \infty$. By Hölder's inequality

$$\begin{gathered} \|Vu\|_{L_t^r L_x^p(S_T)} \leqslant \|V\|_{L^{(n+1)/2}(S_T)} \|u\|_{L_t^s L_x^q(S_T)}, \\ \text{if} \quad \frac{2}{n+1} + \frac{1}{q} = \frac{1}{p}, \ \frac{2}{n+1} + \frac{1}{s} = \frac{1}{r}. \end{gathered} \tag{3.24}$$

The exponents are exactly as in case (i) of the corollary (or equivalenty (3.10′)), so (3.22) with $R = \infty$, follows from (3.19) and (3.24). To prove that the other term in (3.22) is under control we use the Sobolev embedding theorem:

$$\|w(t, \cdot)\|_{L^e(\mathbb{R}^n)} \leqslant C \|w(t, \cdot)\|_{\dot{H}^{n(1/2 - 1/e)}}.$$

But a calculation using (3.10) and (3.10′) gives

$$n\left(\frac{1}{2} - \frac{1}{e}\right) = \frac{n+5}{4} - \frac{n+1}{2p}$$

and hence, by (3.19), the last term is also dominated by $\|Vu\|_{L_t^r L_x^p(S_T)}$. So if we use (3.24) again, we get (3.22) when $R=\infty$.

To prove (3.22) for $R<\infty$ we simply have to note that (3.9) can be localized so that it remains true when we integrate over $S_T \cap \Lambda_{R,0}$ instead of $S_T=[0, T] \times \mathbb{R}^n$. In fact, if $w$ is the solution of $\square w=F$ with vanishing data and $w_\chi$ is the solution of $\square w_\chi=\chi F$ with vanishing data, where $\chi$ is the characteristic function of $S_T \cap \Lambda_{R,0}$, then it follows that $w=w_\chi$ in $S_T \cap \Lambda_{R,0}$.

To prove (3.23) we first note that $\square\, |D_x|^{\gamma-1/2} w=|D_x|^{\gamma-1/2} F(u)$. Now (3.23) follows from (3.20) and the so-called Leibnitz's rule for fractional derivatives (See [3], [4], [6], [17]):

$$\begin{aligned} &\| \, |D_x|^{\gamma-1/2} F(u)\|_{L^{2(n+1)/(n+3)}(S_T)} \\ &\quad \leqslant C \, \|F'(u)\|_{L^{(n+1)/2}(S_T)} \, \| \, |D_x|^{\gamma-1/2} u\|_{L^{2(n+1)/(n-1)}(S_T)} \end{aligned} \tag{3.25}$$

provided that the second condition in (1.2) holds. ∎

*Proofs*

Most of the estimates for the homogeneous Cauchy problem in Theorems 3.1 are well known. First of all, inequality (3.7) is just the well known energy inequality for the linear Cauchy problem. Also, (3.4) is a direct consequence of the Strichartz-1/2 estimate [39] (see [31, p. 215]). So we need only prove (3.5) and (3.6). But, as we shall see, these will follow from the proof of (3.12) and (3.18), respectively.

The inhomogeneous estimates are much harder. Their proof will be based on certain dyadic estimates for $e^{it\sqrt{-\Delta}}$ in $\mathbb{R}^n$. Before stating these, we recall that the solution to the inhomogeneous wave equation with zero data, (3.3), is given by

$$w(t, x)=(2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin(t-s)\,|\xi|}{|\xi|} \hat{F}(s, \xi)\, d\xi\, ds, \tag{3.26}$$

if, for a given fixed $s$, $\hat{F}(s, \xi)$ denotes the $n$-dimensional Fourier transform of $x \to F(s, x)$.

To state the dyadic estimates we need to introduce a bump function. Specifically, let $\beta \in C_0^\infty(\mathbb{R}_+)$ satisfy

$$\sum_{j=-\infty}^{\infty} \beta(\tau/2^j) \equiv 1, \qquad \tau>0.$$

Then, if $\alpha<n$, we define an operator sending functions of $n$-variables to functions of $(n+1)$-variables by setting

$$(W_j^\alpha f)(t, x)=(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi+it|\xi|} \beta(|\xi|/2^j)\, \hat{f}(\xi) \frac{d\xi}{|\xi|^\alpha}, \tag{3.27}$$

where here $\hat{f}$ denote the usual Fourier transform of the function of $n$-variables $f$.

With this notation, the key ingredient in the proof of Theorems 3.2 and 3.3 is the following

PROPOSITION 3.6. *Let $n \geqslant 2$. Then if $1 \leqslant p \leqslant 2$ and if $p'$ denotes the conjugate exponent, $p' = p/(p-1)$, $1 < p \leqslant 2$, and $p' = \infty$, if $p = 1$,*

$$\| W_j^\alpha f \|_{L^{p'}(\mathbb{R}^n)} \leqslant C \lambda^{(2/p-1)(n+1)/2-\alpha} (|t| + \lambda^{-1})^{-(2/p-1)(n+1)/2} \|f\|_{L^p(\mathbb{R}^n)},$$
$$\lambda = 2^j. \tag{3.28}$$

This inequality is well known and easy to prove. If $p = 2$ it just follows from Plancherel's theorem and the fact that $\beta(|\xi|/\lambda)\,|\xi|^{-\alpha}) = O(\lambda^{-\alpha})$. To prove the estimate for $p = 1$, we need to recall that the kernel, $K_j^\alpha(t, x; y)$, of $W_j^\alpha$ satisfies the estimates

$$|K_j^\alpha(t, x; y)| \leqslant C_{N,n,\alpha} \lambda^{(n+1)/2-\alpha} (|t| + \lambda^{-1})^{-(n-1)/2} (1 + \lambda\, |\, |x-y| - |t|\, |)^{-N},$$
$$N = 0, 1, 2, ..., \lambda = 2^j, \tag{3.29}$$

(see [31, pp. 223–224]). We shall need the estimate corresponding to large $N$ later, but to prove (3.28) for $p = 1$, we just use (3.29) with $N = 0$. If we interpolate between the estimates for $p = 1$ and $p = 2$ we get (3.28).

In addition to these dyadic estimates, we shall need a couple of lemmas. The first will allow us to bound operators from $(n+1)$-variables to $(n+1)$-variables using certain estimates for related operators from $n$-variables to $n$-variables.

LEMMA 3.7. *Let $K(t, x; s, y)$ be a measurable function on $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$ and set*

$$(Tf)(t, x) = \int_{\mathbb{R}^{1+n}} K(t, x; s, y)\, f(s, y)\, ds\, dy.$$

*For fixed $s$ and $t$ define the frozen operator*

$$(T_{s,t}\, g)(x) = \int_{\mathbb{R}^n} K(t, x; s, y)\, g(y)\, dy,$$

*and suppose that*

$$\|T_{s,t}\, g\|_{L^q(\mathbb{R}^n)} \leqslant C_0\, |t-s|^{-1+(1/r_1 - 1/r_2)} \|g\|_{L^2(\mathbb{R}^n)}.$$

*Then, if $1 < r_1 < r_2 < \infty$, and if $C_{r_1, r_2}$ is the constant in the one-dimensional $L^{r_1}(\mathbb{R}) \to L^{r_2}(\mathbb{R})$ Hardy-Littlewood inequality for fractional integrals,*

$$\|Tf\|_{L_t^{r_2} L_x^q(\mathbb{R}^{1+n})} \leqslant C_0\, C_{r_1, r_2} \|f\|_{L_t^{r_1} L_x^p(\mathbb{R}^{1+n})}.$$

*Also, if*

$$\|T_{s,t}g\|_{L^q(\mathbb{R}^n)} \leqslant C_0\, C(s,t)\, \|g\|_{L^p(\mathbb{R}^n)},$$

*where $C(s,t)$ satisfies the $L^1$-estimates*

$$\sup_{s\in\mathbb{R}} \int_{-\infty}^{\infty} C(s,t)\,dt,\ \sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} C(s,t)\,ds \leqslant C_1,$$

*and if $1 \leqslant r \leqslant \infty$ if follows that*

$$\|Tf\|_{L^r_t L^q_x(\mathbb{R}^{1+n})} \leqslant C_0\, C_1\, \|f\|_{L^r_t L^p_x(\mathbb{R}^{1+n})}.$$

*Proof.* If we apply Minkowski's integral inequality we get

$$\|Tf\|_{L^{r_2}L^q} \leqslant \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |(T_{t,s}f(s,\cdot))(x)|^q\,dx\right)^{1/q} ds\right)^{r_2} dt\right)^{1/r_2}.$$

If we use our assumption about the frozen operator, we conclude that the last term is

$$\leqslant C_0 \left(\int \left(\int \|f(s,\cdot)\|_{L^p(\mathbb{R}^n)}\, |t-s|^{-1+(1/r_1-1/r_2)}\,ds\right)^{r_2} dt\right)^{1/r_2}.$$

Finally, if we apply the Hardy-Littlewood inequality, we conclude that the last expression is

$$\leqslant C_0\, C_{r_1,r_2} \left(\int \|f(t,\cdot)\|^{r_1}_{L^p(\mathbb{R}^n)}\,dt\right)^{1/r_1} = C_0\, C_{r_1,r_2}\, \|f\|_{L^{r_1}L^p(\mathbb{R}^{1+n})}.$$

The proof of the other inequality in the lemma is similar. ∎

The other lemma we shall need is a simple consequence of Littlewood-Paley theory.

LEMMA 3.8. *If $\beta \in C_0^\infty(\mathbb{R}_+)$ is as above, let us define Littlewood-Paley operators*

$$G_j(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\beta(|\xi|/2^j)\,\hat{G}(t,\xi)\,d\xi,$$

*where, for a given $t$, $\hat{G}(t,\xi)$ denotes the $n$-dimensional Fourier transform of $x \to G(t,x)$. Then*

$$\|G\|_{L^s_t L^q_x} \leqslant C\left(\sum_{j=-\infty}^{\infty} \|G_j\|^2_{L^s_t L^q_x}\right)^{1/2}, \qquad \text{if} \quad 2 \leqslant q < \infty, \quad \text{and} \quad 2 \leqslant s \leqslant \infty,$$

*and*

$$\left(\sum_{j=-\infty}^{\infty} \|G_j\|^2_{L_t^r L_x^p}\right)^{1/2} \leqslant C\,\|G\|_{L_t^r L_x^p}, \qquad \textit{if} \quad 1 < p \leqslant 2, \quad \textit{and} \quad 1 \leqslant r \leqslant 2.$$

*Proof.* The proof relies on Littlewood-Paley theory. Specifically, we shall use the fact that if $1 < p < \infty$ there is a constant $0 < C = C_p < \infty$ so that

$$C^{-1} \leqslant \left\| \left(\sum |G_j(t, \cdot)|^2\right)^{1/2} \right\|_{L_x^p} \Big/ \|G(t, \cdot)\|_{L_x^p} \leqslant C.$$

For a proof of this see e.g. [31]. The constants in the inequalities in the lemma are just this constant $C$.

To see this, let us focus on the first inequality. If we use the first Littlewood-Paley inequality, we get that

$$\|G(t, \cdot)\|^2_{L_x^q} \leqslant C\left(\int_{\mathbb{R}^n} \left(\sum |G_j(t, x)|^2\right)^{q/2} dx\right)^{2/q}.$$

However, since $q/2 \geqslant 1$, Minkowski's inequality implies that the last expression is

$$\leqslant C \sum \|G_j(t, \cdot)\|^2_{L_x^q}.$$

If we now use this and the fact that $s/2 \geqslant 1$, we similarly get

$$\begin{aligned} \|G\|^2_{L_t^s L_x^q} &= \left(\int \|G(t, \cdot)\|^{2 \cdot s/2}_{L_x^q} dt\right)^{2/s} \\ &\leqslant C\left(\int \left(\sum \|G_j(t, \cdot)\|^2_{L_x^q}\right)^{s/2} dt\right)^{2/s} \\ &= C \sum \|G_j\|^2_{L_t^s L_x^q}. \end{aligned}$$

The proof of the other inequality uses the second Littlewood-Paley inequality and is similar. ∎

Having presented the main tools, let us turn to the proof of the estimates for the inhomogeneous wave equation. If we recall (3.26) and take imaginary parts, we conclude that, in order to prove any of these estimates, it suffices to prove the analogous one where, if $\alpha = 1$, $w$ is replaced by

$$(T^\alpha F)(t, x) = (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} e^{ix \cdot \xi + i(t-s)|\xi|} \hat{F}(s, \xi) \frac{d\xi}{|\xi|^\alpha}\, ds. \tag{3.30}$$

Additionally, if we note that the partial Fourier transform of $|D_x|^\gamma F$ is $|\xi|^\gamma \hat{F}(s, \xi)$, we can simplifiy the inequalities involving fractional derivatives and homogeneous Sobolev norms, if we also recall that $|D_x|^\gamma$ commutes with $\Box$.

Let us be more specific and start with (3.8):

*Proof of* (3.8). Notice that, in view of the above remarks, this inequality would follow from showing that

$$\|T^\alpha F\|_{L^q(\mathbb{R}^{1+n})} \leq C_q \|F\|_{L^{2(n+1)/(n+3)}}, \qquad \text{if } \frac{2(n+1)}{n-1} \leq q < \infty$$
$$\text{and} \quad \alpha = (n+1)\left(\frac{1}{2} - \frac{1}{q}\right). \tag{3.8'}$$

If we use the Littlewood-Paley lemma, Lemma 3.8, we notice that since $2(n+1)/(n+3) < 2 < q < \infty$, it suffices to prove the dyadic version of (3.8′):

$$\|T_j^\alpha F\|_{L^q(\mathbb{R}^{1+n})} \leq C_q \|F\|_{L^{2(n+1)/(n+3)}}, \qquad \text{if } \frac{2(n+1)}{n-1} \leq q \leq \infty$$
$$\text{and} \quad \alpha = (n+1)\left(\frac{1}{2} - \frac{1}{q}\right), \tag{3.8''}$$

if

$$(T_j^\alpha F)(t, x) = (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} e^{ix\cdot\xi + i(t-s)|\xi|} \beta(|\xi|/\lambda)\, \hat{F}(s, \xi)\, \frac{d\xi}{|\xi|^\alpha}\, ds.$$

To see that (3.8″) implies (3.8′), we note that there is a constant $c_0$, depending on $\beta$, for which $T_j^\alpha F = T_j^\alpha(\sum_{|j-k| \leq c_0} F_k)$. Thus, if (3.8″) held, Lemma 3.8 would yield

$$\|T^\alpha F\|_{L^q}^2 \leq C \sum_j \|T_j^\alpha F\|_{L^q}^2 \leq C' \sum_j \sum_{|j-k| \leq c_0} \|T_j^\alpha F_k\|_{L^q}^2$$
$$\leq C'' \sum_j \sum_{|j-k| \leq c_0} \|F_k\|_{L^{2(n+1)/(n+3)}}^2,$$

and since another application of Lemma 3.8 implies that the last expression is dominated by the square of the right side of (3.8′), we have proved our claim.

Notice that, unlike (3.8′), (3.8″) holds for $q = \infty$, and in fact (3.8″) would follow, by interpolation, if we could prove the special cases where $q = 2(n+1)/(n-1)$ and $q = \infty$.

To prove the $L^\infty$ estimate in (3.8″), we notice that, by Hölder's inequality, it suffices to show that the kernel $K_j^\alpha$ satisfies

$$\sup_{(t,x)} \int_{\mathbb{R}^{1+n}} |K_j^\alpha(t, x; s, y)|^{2(n+1)/(n-1)} \, ds \, dy < \infty, \qquad \alpha = \frac{n+1}{2}.$$

But if we use (3.29) we can dominate the left side by

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} (\lambda^{-1} + |s|)^{-(n-1)/2 \cdot 2(n+1)/(n-1)} \cdot (1 + \lambda \, ||y| - |s||)^{-N} \, dy \, ds$$

$$\leqslant C \int_{-\infty}^{\infty} (\lambda^{-1} + |s|)^{-(n+1)} \lambda^{-1}(\lambda^{-1} + |s|)^{n-1} \, ds$$

$$= C \int_{-\infty}^{\infty} \lambda^{-1}(\lambda^{-1} + |s|)^{-2} \, ds = C',$$

assuming that $N > n$.

The other endpoint in (3.8″), where $q = 2(n+1)/(n-1)$, is well known (cf. [39]); however, for the sake of completeness let us present the proof. It will be based on the following lemma which we state in full generality for later use.

LEMMA 3.9. *Let $n \geqslant 2$ and assume that*

$$\max\left\{\frac{2(n-1)}{n+1}, 1\right\} < p \leqslant 2.$$

*For a given p as above, define*

$$r = r(p) = \frac{4p}{(n+3)p - 2(n-1)}, \tag{3.31}$$

*and let $r'$ denote the conjugate exponent. Then*

$$\|T_j^\alpha F\|_{L_t^{r'} L_x^{p'}(\mathbb{R}^{1+n})} \leqslant C_{\alpha, p} \lambda^{(2/p-1)(n+1)/2 - \alpha} \|F\|_{L_t^r L_x^p(\mathbb{R}^{1+n})}. \tag{3.32}$$

*Moreover,*

$$\|T\alpha F\|_{L_t^{r'} L_x^{p'}(\mathbb{R}^{1+n})} \leqslant C_p \|F\|_{L_t^r L_x^p(\mathbb{R}^{1+n})}, \qquad \text{if} \quad \alpha = \frac{n+1}{2}\left(\frac{2}{p} - 1\right). \tag{3.33}$$

To see how (3.32) implies the other endpoint in (3.8″) notice that

$$r(p) = p \Leftrightarrow p = \frac{2(n+1)}{n+3},$$

and that

$$0=\frac{n+1}{2}\left(\frac{2}{p}-1\right) \quad \text{if} \quad p=\frac{2(n+1)}{n+3},$$

and

$$\alpha=\frac{n+1}{2}-(n+1)\frac{n-1}{2(n+1)}=1.$$

On account of this, we deduce that the special case of (3.8″) where $q=2(n+1)/(n-1)$ is actually in turn a special case of (3.32). Hence the proof of (3.8) will be complete when we have established the lemma. ∎

*Proof of Lemma* 3.9. For the sake of notation let us fix $\alpha$ and $\lambda=2^j$. We then set $T=T_j^\alpha$ and, in the notation of Lemma 3.7, define the associated frozen operators, sending functions of $n$-variables to functions of $n$-variables, by

$$(T_{s,t}\,g)(x)=\int_{\mathbb{R}^n} K_j^\alpha(t, x; s, y)\, g(y)\, dy.$$

If we use Proposition 3.6 we get the following estimates for these operators:

$$\|T_{s,t}\,g\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\lambda^{(2/p-1)(n+1)/2-\alpha}\, |t-s|^{-(2/p-1)(n-1)/2}\, \|g\|_{L^p(\mathbb{R}^n)}. \qquad (3.32')$$

But, if $\max\{2(n-1)/(n+1), 1\}<p\leqslant 2$, it follows that $1\leqslant r(p)<2$, and, also, a straightforward calculation shows that

$$\frac{n-1}{2}\left(\frac{2}{p}-1\right)=1-\left(\frac{1}{r(p)}-\frac{1}{r(p)'}\right).$$

Hence (3.32′) and Lemma 3.7 yield (3.32).

Inequality (3.33) follows from (3.32) and an application of Lemma 3.8, as in the proof of (3.8′). ∎

*Proof of* (3.12). We start out by observing that, using the above notation, (3.12) would follow from showing that we have uniform bounds

$$\|(T^\alpha F)(t_0, \cdot)\|_{L^2(\mathbb{R}^n)} \leqslant C_p\, \|F\|_{L_t^r L_x^p(\mathbb{R}^{1+n})}, \qquad t_0\in\mathbb{R}, \qquad (3.12')$$

if $\max\{2(n-1)/(n+1), 1\}<p\leqslant 2$, $r$ is as in (3.31) and

$$\alpha=-1+\frac{n+5}{4}-\frac{n+1}{2p}=\frac{n+1}{4}\left(\frac{2}{p}-1\right).$$

But if we use the first step in the proof of the $L^2$-restriction theorem we get

$$\int_{\mathbb{R}^n} |(T^\alpha F)(t_0, x)|^2 \, dx = \int_{\mathbb{R}^{1+n}} (T^{2\alpha} F)(t, x) \cdot \overline{F(t, x)} \, dt \, dx$$
$$\leqslant \|T^{2\alpha} F\|_{L_t^{r'} L_x^{p'}} \cdot \|F\|_{L_t^r L_x^p}.$$

Thus, since

$$2\alpha = \frac{n+1}{2} \left( \frac{2}{p} - 1 \right),$$

(3.12′) now follows from Lemma 3.9. ∎

*Proof of* (3.5). Since

$$v(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{\sin t\,|\xi|}{|\xi|} \hat{f}(\xi) \, d\xi$$
$$+ (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \cos(t\,|\xi|) \, \hat{g}(\xi) \, d\xi,$$

we see that it suffices to prove that if

$$(W^\alpha f)(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi + it\,|\xi|} \hat{f}(\xi) \frac{d\xi}{|\xi|^\alpha},$$

then, for $2 \leqslant q < \infty$, $n = 2, 3$, or $2 \leqslant q < 2(n-1)/(n-3)$, $n \geqslant 4$,

$$\|W^\alpha f\|_{L_t^s L_x^q(\mathbb{R}^{1+n})} \leqslant C \, \|f\|_{L^2(\mathbb{R}^n)}, \qquad \alpha = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right). \tag{3.5′}$$

But the dual version of this is

$$\|(W^\alpha)^* F\|_{L^2(\mathbb{R}^n)} \leqslant C_p \, \|F\|_{L_t^r L_x^p(\mathbb{R}^{1+n})}, \tag{3.5″}$$

for $p$ and $r$ as in (3.12′). Since the adjoint of $W^\alpha$ is essentially the frozen operator in (3.12′), the proof of this inequality gives us (3.5″)

*Proof of* (3.11). This just follows trivially from (3.12), because

$$|D_x|^\gamma F = |D_x|^\gamma \square w = \square(|D_x|^\gamma w)$$

and because (3.13) equals 1/2 when $p = 2(n+1)/(n+3)$. ∎

*Proof of* (3.9). It suffices to show that

$$\|T^1 F\|_{L_t^s L_x^q} \leqslant C_p \, \|F\|_{L_t^r L_x^p},$$

if $p$, $r$, $s$ and $q$ are as in (3.10). By duality, we can assume that $\max\{2(n-1)/(n+1), 1\} < p \leqslant 2(n+1)/(n+3)$, since the inequality for $2(n+1)/(n+3) \leqslant p < 2(n+1)(n-1)/((n+1)^2-8)$, $n \geqslant 3$, $6/5 \leqslant p < 3/2$, $n=2$, just involves exponents which are dual to the ones occurring under the above restriction. Since $r, p < 2 < s, q$, we can apply Lemma 3.8, as in the proof of (3.8′), to see that this is equivalent to showing that we have uniform dyadic estimates:

$$\|T_j^1 F\|_{L_t^s L_x^q} \leqslant C_p \|F\|_{L_t^r L_x^p}, \quad j \in \mathbb{Z}, \qquad \max\left\{\frac{2(n-1)}{n+1}, 1\right\} < p \leqslant \frac{2(n+1)}{n+3}. \tag{3.9′}$$

But (3.32) gives

$$\|T_j^1 F\|_{L_t^{r'} L_x^{p'}} \leqslant C_p \lambda^{(2/p-1)(n+1)/2-1} \|F\|_{L_t^r L_x^p}, \qquad \lambda = 2^j,$$

while (3.33) and Plancherel's theorem imply that

$$\|T_j^1 F\|_{L_t^\infty L_x^2} \leqslant C_p \lambda^{(2/p-1)(n+1)/4-1} \|F\|_{L_t^r L_x^p}, \qquad \lambda = 2^j.$$

These two estimates and interpolation yield (3.9′). This is because

$$0 = \theta \cdot \left[\frac{n+1}{2}\left(\frac{2}{p}-1\right)-1\right] + (1-\theta)\cdot\left[\frac{n+1}{4}\left(\frac{2}{p}-1\right)-1\right],$$
$$\text{if} \quad \theta = \frac{(n+5)p-2(n+1)}{(n+1)(2-p)},$$

and, for this $\theta$,

$$\frac{1}{s} = \frac{1}{r'}\,\theta + \frac{1}{\infty}\,(1-\theta) \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{p'}\,\theta + \frac{1}{2}\,(1-\theta),$$

if $s$ and $q$ are as above. ∎

To finish the proof of the inequalities we just have to prove the inequalities in higher dimensions corresponding to $2n/(n+3) \leqslant p < 2(n-1)/(n+1)$. We start out with the estimate involving $L^2$ since it will be used in the proof of the mixed norm estimates for small $p$.

*Proof of* (3.18), We follow the proof of (3.12). If we use Lemma 3.8, as in the proof of (3.8′), and now let

$$\alpha = -1 + n\left(\frac{n+3}{2n} - \frac{1}{p}\right) = \frac{n+1}{2} - \frac{n}{p},$$

it suffices to show that we have the following dyadic estimates

$$\|(T_j^\alpha F)(t_0,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant C_p \|F\|_{L_t^2 L_x^p(\mathbb{R}^{1+n})}, \qquad j\in\mathbb{Z}, \quad t_0\in\mathbb{R}, \tag{3.18'}$$

assuming as above that $2n/(n+3)\leqslant p<2(n-1)/(n+1)$. As in the proof of (3.12), this in turn would follow from the related estimates

$$\|T_j^{2\alpha} F\|_{L_t^2 L_x^{p'}(\mathbb{R}^{1+n})} \leqslant C_p \|F\|_{L_t^2 L_x^p(\mathbb{R}^{1+n})}. \tag{3.18''}$$

But, using the notation of Lemma 3.7, we see that Proposition 3.6 gives the following bounds for the associated frozen operators

$$\|T_{j,s,t}^{2\alpha} g\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\lambda^{(2/p-1)(n+1)/2-2\alpha}(\lambda^{-1}+|t-s|)^{-(2/p-1)(n-1)/2}$$
$$\times \|g\|_{L^p(\mathbb{R}^n)}, \qquad \lambda=2^j.$$

Notice that for this range of $p$,

$$\frac{n-1}{2}\left(\frac{2}{p}-1\right)>1,$$

and, consequently,

$$\int_{-\infty}^{\infty} (\lambda^{-1}+|t-s|)^{-(2/p-1)(n-1)/2}\,ds = C_p\lambda^{(2/p-1)(n-1)/2-1}.$$

Hence (3.18″) and Lemma 3.7 yield

$$\begin{aligned}\|T_j^{2\alpha} F\|_{L_t^2 L_x^{p'}(\mathbb{R}^{1+n})} &\leqslant C\lambda^{(2/p-1)(n+1)/2-2\alpha}\cdot\lambda^{(2/p-1)(n-1)/2-1}\|F\|_{L_t^2 L_x^p(\mathbb{R}^{1+n})}\\ &= C\lambda^{2n/p-(n+1)-2\alpha}\|F\|_{L_t^2 L_x^p(\mathbb{R}^{1+n})}\\ &= C\|F\|_{L_t^2 L_x^p(\mathbb{R}^{1+n})},\end{aligned}$$

finishing the proof of (3.18′) and hence (3.18). ∎

*Proof of* (3.6). Let $W^\alpha$ be as in (3.5′). Then if $W_j^\alpha$ are the associated dyadic operators, it suffices to show that, when $n\geqslant 4$ and $2(n-1)/(n-3)<q\leqslant 2n/(n-3)$ one has uniform estimates

$$\|W_j^\alpha f\|_{L_t^s L_x^q(\mathbb{R}^{1+n})} \leqslant C\|f\|_{L^2(\mathbb{R}^n)}, \qquad \alpha=\frac{n-1}{2}-\frac{n}{q}, \quad j\in\mathbb{Z}. \tag{3.6'}$$

But the adjoint of $W_j^\alpha$ is essentially the frozen operator in (3.18′) with $t_0=0$. Hence (3.18′) is essentially the dual version of (3.6′) and so its proof yields (3.6′). ∎

*Proof of* (3.14) *and* (3.16). We may assume in (3.14) that $2n/(n+3)<p<2(n-1)/(n+1)$ because the inequality for $p=2n/(n+3)$ is a special case of (3.18). Proving (3.14) for the remaining range of $p$ is the most difficult part of the proof of Theorem 3.3.

We shall follow the argument used to prove (3.9); however, we shall need to make an additional decomposition of the operators arising and will interpolate between inequalities involving different mixed-norms.

As before, (3.14) and (3.16) follow from the appropriate mixed-norm boundedness of $T^1$. By Lemma 3.8 and the proof that (3.8″) implies (3.8′), it suffices to prove the uniform dyadic estimates

$$\|T^1_\lambda F\|_{L^s_t L^q_x} \leqslant C_p \|F\|_{L^2_t L^p_x}, \qquad \lambda = 2^j, \quad j \in \mathbb{Z}, \tag{3.14'}$$

if $p$ is as above and that $q$ and $s$ are given by (3.18), since the dual version of this (along with the dual version of (3.18′) corresponding to $q = 2n/(n-3)$) yields (3.16).

For the sake of notation, let us fix $\lambda = 2^j$ and define for $k = 1, 2, \ldots$

$$\begin{aligned}(S_k F)(t, x) = (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} & e^{ix\cdot\xi + i(t-s)|\xi|} \beta(|\xi|/\lambda)\, \beta(\lambda 2^{-k}(t-s)) \\ & \times \hat{F}(s, \xi) \frac{d\xi}{|\xi|}\, ds,\end{aligned}$$

where $\beta \in C_0^\infty(\mathbb{R}_+)$ is as above. Thus, if we let

$$S_0 F = T^1_\lambda F - \sum_{k=1}^{\infty} S_k F,$$

it suffices to show that

$$\|S_k F\|_{L^s_t L^q_x} \leqslant C 2^{-\varepsilon_p k} \|F\|_{L^2_t L^p_x}, \tag{3.34}$$

for some $\varepsilon_p > 0$.

Notice that since we have the scaling condition

$$n\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{s} - \frac{1}{2} = 2,$$

in proving (3.34), it suffices to prove the special case where $\lambda = 1$, since the other dyadic estimates will follow from this and a change of variables. Thus, for the sake of notation, we shall assume that $\lambda = 1$ in what follows.

A key observation that will be need for the proof of (3.34) is that

$$F(t, x) \equiv 0, \qquad t \notin I \Rightarrow (S_k F)(t, x) \equiv 0, \qquad t \notin I^*, \tag{3.35}$$

if $I$ is an interval of length $2^k$ and $I^*$ is the interval with the same center but sidelength $C_0 2^k$, with $C_0$ being a fixed constant depending only on $\beta$.

This assertion just follows from the fact that, by construction, the kernel of $S_k$ vanishes unless $t-s \approx 2^k$.

In addition to (3.35) we shall need two inequalities. First, if we repeat the arguments that we used to prove (3.18) we get that

$$\begin{aligned}\|S_k F\|_{L_t^2 L_x^{p'}} &\leqslant C(2^k)^{-(2/p-1)(n-1)/2+1} \|F\|_{L_t^2 L_x^p} \\ &= C2^{-k((n-1)/p-(n+1)/2)} \|F\|_{L_t^2 L_x^p}.\end{aligned} \tag{3.36}$$

Here, as in the proof of (3.18), we are using the fact that

$$\frac{n-1}{2}\left(\frac{2}{p}-1\right) > 1,$$

and hence the bounds in (3.36) become worse and worse as $k$ decreases. (This accounts for the difficulty of the proof of (3.14) and the fact that this inequality is worse than (3.9).) In addition to (3.36), we shall need

$$\|S_k F\|_{L_t^\infty L_x^2} \leqslant C \|F\|_{L_t^2 L_x^p}, \tag{3.37}$$

which is an immediate consequence of (3.18).

We now have all the tools needed to prove (3.34) and hence (3.14). The first step is to interpolate between (3.36) and (3.37). A calculation shows that if $q$ is as above then

$$\frac{4n-(n+5)p}{2(n+1)p} = \frac{1}{q} = \theta \frac{1}{2} + (1-\theta)\frac{1}{p'} \Leftrightarrow \theta = \frac{6n+2-(3n+7)p}{(n+1)(2-p)}.$$

If we then let $s_0 = s_0(p)$ be defined by

$$\frac{1}{s_0} = \theta \frac{1}{\infty} + (1-\theta)\frac{1}{2} = \frac{(n+3)p-2n}{(n+1)(2-p)},$$

we can interpolate between (3.36) and (3.37) to obtain

$$\|S_k F\|_{L_t^{s_0} L_x^q} \leqslant C2^{-k((n-1)/p-(n+1)/2)(1-\theta)} \|F\|_{L_t^2 L_x^p}.$$

However, a straightforward calculation shows that

$$\left(\frac{n-1}{p}-\frac{n+1}{2}\right)(1-\theta) = 2\left(\frac{1}{s}-\frac{1}{s_0}\right).$$

Thus the last inequality can be rewritten as

$$\|S_k F\|_{L_t^{s_0} L_x^q} \leqslant C2^{-2k(1/s-1/s_0)} \|F\|_{L_t^2 L_x^p}. \tag{3.38}$$

We claim that, since $s<s_0$, this inequality, along with (3.35) and Hölder's inequality yields (3.34) with

$$\varepsilon_p = \frac{1}{s} - \frac{1}{s_0}.$$

To see this, let $\{I_l\}$ be a partition of $\mathbb{R}$ into nonoverlapping intervals of length $2^k$, and let $\chi_{I_l}(t)$ be the characteristic function of $I_l$. Then since the expanded intervals $I_l^*$ have uniformly finite overlap (3.35) gives

$$\|S_k F(t,\cdot)\|^s_{L^q_x} \leqslant C_s \sum_l \|S_k(\chi_{I_l} F)(t,\cdot)\|^s_{L^q_x},$$

with $C_s$ being a constant depending only on $s$ and $C_0$. If we use (3.35) again we get

$$\|S_k F\|^s_{L^s_t L^q_x} \leqslant C \sum_l \|S_k(\chi_{I_l} F)\|^s_{L^s_t L^q_x(I_l^* \times \mathbb{R})}.$$

Since $|I_l^*| = C_0/2^k$, if we apply Hölder's inequality and then (3.38) we get

$$\|S_k F\|^s_{L^s_t L^q_x} \leqslant C 2^{-ks(1/s - 1/s_0)} \sum_l \|\chi_{I_l} F\|^s_{L^2_t L^p_x}.$$

But $2<s$ and hence, since the $\chi_{I_l} F$ have nonoverlapping supports

$$\sum_l \|\chi_{I_l} F\|^s_{L^2_t L^p_x} \leqslant \|F\|^s_{L^2_t L^p_x}.$$

Combining the last two inequalities yields (3.34) with $\varepsilon_p$ as above. ∎

This completes the proof of the proof of the inequalities.

## 4. Existence Proofs

In this section we shall prove the existence results in Theorems 2.1–2.3. The uniqueness results will be handled in the next section.

Let us now outline the the main steps of the argument for $\gamma > 1/2 - 1/(n-1)$, $n \geqslant 3$, or $\gamma > 1/4$, $n=2$ (or equivalently $\kappa > \kappa_0$, where $\kappa_0 = (n+1)^2/((n-1)^2 + 2^2)$, $n \geqslant 3$ and $\kappa_0 = 3$, $n=2$.) This is the range of powers included in Theorem 2.1. Let

$$S_T = \{(t,x) : 0<t<T\}, \tag{4.1}$$

and let $F_k$ satisfy (1.2) with $\kappa$ as above, i.e., $F_\kappa(u) \approx |u|^\kappa$. Then, if

$$f \in \dot{H}^\gamma(\mathbb{R}^n) \text{ and } g \in \dot{H}^{\gamma-1}(\mathbb{R}^n),$$

$$\text{with } \gamma = \begin{cases} \dfrac{n+1}{4} - \dfrac{1}{\kappa-1}, & \text{if } \kappa_0 < \kappa \leqslant \dfrac{n+3}{n-1} \\ \dfrac{n}{2} - \dfrac{2}{\kappa-1}, & \text{if } \dfrac{n+3}{n-1} \leqslant \kappa \leqslant \dfrac{n+1}{n-3}, \end{cases} \tag{4.2}$$

are fixed, we shall show that there is a unique weak solution to the Cauchy problem

$$\begin{cases} \Box u = F_\kappa(u) \\ u(0, x) = f(x), \qquad \partial_t u(0, x) = g(x) \end{cases} \tag{4.3}$$

in $S_{T_*}$ for some $T_* > 0$, having the properties that

$$(u, \partial_t u) \in L^\infty([0, T_*]; \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)), \tag{4.4}$$

$$u \in L^{(\kappa-1)(n+1)/2}(S_{T_*}), \tag{4.5}$$

as well as

$$u \in L_t^s L_x^q(S_{T_*}), \qquad s = \frac{4q}{(n-1)(q-2)}, \qquad \text{and}$$

$$q = \frac{2(n+1)}{n+1-4\gamma} \qquad \text{if} \quad \left|\gamma - \frac{1}{2}\right| < \frac{1}{n-1} \tag{4.6}$$

$$|D_x|^{\gamma-1/2}\, u \in L^{2(n+1)/(n-1)}(S_{T_*}),$$

$$q = \frac{2(n+1)}{n-2\gamma}, \qquad \text{if} \quad \frac{1}{2} \leqslant \gamma \leqslant \frac{3}{2}.$$

As before, if $n = 2$, we need to replace the first condition on $\gamma$ with $|\gamma - 1/2| < 1/4$. The proof of (4.4) will also show that $(u, \partial_t u) \in C([0, T_*]; \dot{H}^\gamma \times \dot{H}^{\gamma-1})$.

The norms occurring in (4.6) are substitutes for the norm in (4.4). These behave better under the nonlinear mapping (see Proposition 3.5). The norms in (4.6) scale in the same way as the norm in (4.4), and the spaces in (4.6) comes up naturally when we want to construct a solution by iteration as in (4.8). Then, once we found a solution of (4.3) in the spaces in (4.6), with data as in (4.2), it is easy to deduce that the solution has to be in the space (4.4) as well.

The way we get a restriction on $\kappa$ for a fixed $\gamma$ is that we need the solution to be in the space in (4.5) in order for the nonlinear mapping to be a contraction in the spaces in (4.6). As we shall see, it is also needed for uniqueness. Therefore, we need the conditions (4.6) (together with (4.4)) to be strong enough to ensure that the solution is in (4.5) as well, and this gives the restriction (4.2) on $\kappa$.

To find a solution of (4.3) we shall use standard iteration arguments, making use of the estimates in Section 3. Specifically, we shall set

$$u_{-1} \equiv 0 \tag{4.7}$$

and define $u_m$, $m = 0, 1, 2, \ldots$, by

$$\begin{gathered} \square u_m = F_\kappa(u_{m-1}), \\ u_m(0, x) = f(x), \qquad \partial_t u_m(0, x) = g(x). \end{gathered} \tag{4.8}$$

Then, we need to show that there is $T_* > 0$ and function $u$ verifying (4.4) and (4.5) and having the property that

$$u_m \to u, \qquad \text{and} \qquad F_\kappa(u_m) \to F(u), \qquad \text{in } \mathscr{D}'(S_{T_*}). \tag{4.9}$$

The condition (4.9) of course implies that the limit $u$ will be a weak solution of the equation (4.3). The main step in proving (4.9) is to show that the nonlinear mapping $u_m \to u_{m+1}$, defined by (4.8) is a contraction in the spaces given by (4.6). This follows directly from Proposition 3.5 from which we also see that we must have

$$\|F'_\kappa(u_m)\|_{L^{(n+1)/2}(S_T)} \leqslant \frac{1}{2C_q}$$

in order for it to be a contraction. With $F_\kappa \sim |u|^\kappa$ this is true if

$$\|u_m\|_{L^{(\kappa-1)(n+1)/2}(S_T)} \leqslant \varepsilon_0, \tag{4.10}$$

for some $\varepsilon_0 > 0$ and all $m$. The different cases in (4.6) will be dealt with in Lemma 4.1, Lemma 4.2 and Lemma 4.3. But they all follow from Proposition 3.5. One just needs to choose a space for the iteration which is strong enough that it, together with the initial condition, implies (4.10). The interval of $\gamma$ which is covered in Lemma 4.2 is included in Lemma 4.3, but the proof by Lemma 4.2 has many advantages. In particular, it does not use derivatives and Leibnitz's rule for fractional derivatives and hence it can easily be localized to a cone.

*Proof of existence for* $1/2 - 1/(n-1) < \gamma \leqslant 1/2$, $n \geqslant 3$, *or* $1/4 < \gamma \leqslant 1/2$, $n = 2$ (*or equivalently* $\kappa_0 < \kappa < (n+3)/(n-1)$.). Here we shall need the following

LEMMA 4.1. *Assume that* $1/2 - 1/(n-1) < \gamma \leqslant 1/2$ *where* $\gamma = (n+1)/4 - 1/(\kappa - 1)$. *Let* $s = 4q/(n-1)(q-2)$ *and* $q = 2(n+1)/(n+1-4\gamma) = (n+1)/2\ (\kappa - 1)$ *and set*

$$M_m(T) = \|u_m\|_{L_t^s L_x^q(S_T)}, \qquad N_m(T) = \|u_m - u_{m-1}\|_{L_t^s L_x^q(S_T)}.$$

*Then there is an* $\varepsilon_0 > 0$ *so that*

$$M_m(T) \leqslant 2M_0(T),\ N_{m+1}(T) \leqslant \tfrac{1}{2} N_M(T),$$

$$\text{if} \quad 2M_0(T)\, T^{1/(\kappa-1)-(n-1)/4} \leqslant \varepsilon_0, \quad (4.11)$$

*in which case* (4.10) *is satisfied as well.*

*Proof of Lemma* 4.1. The proof uses induction over $m$ and the estimates (3.22) in Proposition 3.5. If we write

$$\Box(u_{m+1} - u_{k+1}) = V_\kappa(u_m, u_k)(u_m - u_k),$$

$$\text{where} \quad V_\kappa(u, v) = \frac{F_\kappa(u) - F_\kappa(v)}{u - v}$$

we see from Proposition 3.5 that

$$\|u_{m+1} - u_{k+1}\|_{L_t^s L_x^q(S_T)} \leqslant \tfrac{1}{2} \|u_m - u_k\|_{L_t^s L_x^q(S_T)}, \qquad (4.12)$$

if

$$\|V_\kappa(u_m, u_k)\|_{L^{(n+1)/2}(S_T)} \leqslant \frac{1}{2C_q}.$$

Since $|V_\kappa(u, v)| \leqslant C(|u| + |v|)^{\kappa - 1}$ we see that there exists an $\varepsilon_0 > 0$ such that this is true if

$$\|u_j\|_{L^{(n+1)/2)(\kappa-1)}(S_T)} \leqslant \varepsilon_0, \qquad j = m, k, \qquad (4.13)$$

which by Hölder's inequality is equivalent to $M_j(T)\, T^{1/(\kappa-1)-(n-1)/4} \leqslant \varepsilon_0$. We want to use induction to prove that $M_m(T) \leqslant 2M_0(T)$. If we choose $k = -1$ in (4.12) we get the first part of (4.11), since $u_{-1} = 0$ and hence

$$M_{m+1} \leqslant M_0 + \tfrac{1}{2} M_m, \qquad \text{if} \quad M_m(T)\, T^{1/(\kappa-1)-(n-1)/4} \leqslant \varepsilon_0.$$

The other part of (4.11) just involves taking $k = m - 1$. ∎

By (3.5),

$$M_0(T) \leqslant C(\|f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}),$$

where the right hand side is bounded by assumption (4.2). Therefore we can choose $T = T_*$ is as in (4.11). Since $u_{-1} = 0$ we have $M_0(T) = N_0(T)$ and it follows from (4.11) that $u_m$ converges to a limit $u$ in $L_t^s L_x^q(S_{T_*})$ and hence in $\mathscr{D}'$. To see that $F_\kappa(u_m) \to F_\kappa(u)$ in $\mathscr{D}'$ it suffices to show that

$$\|F_\kappa(u)\|_{L_t^r L_x^p(S_{T_*})} \leqslant C, \tag{4.14}$$

and

$$\|F_\kappa(u_m) - F_\kappa(u)\|_{L_t^r L_x^p(S_{T_*})} \to 0,$$

if

$$p = \frac{n+1}{2}\,\frac{\kappa-1}{\kappa} = \frac{q}{\kappa}, \qquad \text{and} \qquad r = \frac{4p}{(n+3)p - 2(n-1)}.$$

Notice that this implies that $p$, $r$, $s$, $q$ are as in (3.9). To prove the inequality, since $\kappa r \leqslant s$, we can use (1.2) and Hölder's inequality to get

$$\|F_\kappa(u)\|_{L_t^r L_x^p(S_{T_*})} \leqslant C_0\, \|u\|^\kappa_{L_t^{\kappa r} L_x^q(S_{T_*})} \leqslant C_0\, T_*^{1/r - \kappa/s}\, \|u\|^\kappa_{L_t^s L_x^q(S_{T_*})} \leqslant C,$$

since we just saw that $u \in L_t^s L_x^q(S_{T_*})$. To prove the convergence in $L_t^r L_x^p(S_{T_*})$ we apply Proposition 3.5 as above to get

$$\|F_\kappa(u) - F_\kappa(u_m)\|_{L_t^r L_x^p(S_{T_*})} \leqslant \|V_\kappa(u, u_m)\|_{L^{(n+1)/2}(S_{T_*})}\, \|u - u_{m-1}\|_{L_t^s L_x^q(S_{T_*})}.$$

By the proof of Lemma 4.1, the first factor on the right is $O(1)$, and since the other factor goes to zero, we conclude that $F_\kappa(u_m) \to F_\kappa(u)$ as desired.

Lemma 4.1 of course implies that $u$ must satisfy (4.6) and (4.5), while, (4.4) then follows from (3.19) in Corollary 3.4 with $F = F_\kappa(u)$ satisfying (4.14). ∎

*Proof of existence for* $1/2 \leqslant \gamma < 1/2 + 1/(n-1)$ (*or equivalently* $(n+3)/(n-1) \leqslant \kappa < ((n+1)^2 - 6)/((n-1)^2 - 2)$. In this case we will only give the main points of the proof since this interval is included in the one in Lemma 4.3 below. The argument, however, is used in the propagation of regularity results in the next section.

LEMMA 4.2. *Assume that* $1/2 \leqslant \gamma < 1/2 + 1/(n-1)$ *where* $\gamma = n/2 - 2/(\kappa - 1)$. *Let* $s = 4q/(n-1)(q-2)$, $q = 2(n+1)/(n+1-4\gamma)$ *and* $n/e = n/q + 1/s$ *and set*

$$\begin{aligned} M_m(T) &= \|u_m\|_{L_t^s L_x^q(S_T)}, + \|u_m - u_0\|_{L_t^\infty L_x^e(S_T)}, \\ N_m(T) &= \|u_m - u_{m-1}\|_{L_t^s L_x^q(S_T)} \end{aligned} \tag{4.15}$$

*Then there is an* $\varepsilon_0 > 0$ *so that*

$$M_m(T) \leqslant 2M_0(T), \qquad N_{m+1}(T) \leqslant \tfrac{1}{2} N_m(T),$$
$$\text{if} \quad 2M_0(T)^\alpha \,(2M_0 + \|u_0\|_{L_t^\infty L_x^e(S_T)})^{1-\alpha} \leqslant \varepsilon_0, \tag{4.16}$$

*where* $\alpha > 0$, *in which case* (4.10) *is satisfied as well.*

*Proof of Lemma* 4.2. The proof is exactly the same as in Lemma 4.1. We only need to make sure that (4.13) holds. Now $e$, $(s, q)$ and $\gamma$ are related in such a way that the norms in $L_t^s L_x^q(S_T)$, $L_t^\infty L_x^e(S_T)$ and $\dot{H}^\gamma$ scale in the same way. Hence by Sobolev's lemma:

$$\|u(t, \cdot)\|_{L^e(\mathbb{R}^n)} \leqslant C \,\|u(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)}$$

By Hölder's inequality

$$\|u_m\|_{L^{(\kappa-1)(n+1)/2}(S_T)} \leqslant \|u_m\|^\alpha_{L_t^s L_x^q(S_T)} \,\|u_m\|^{1-\alpha}_{L_t^\infty L_x^e(S_T)}, \tag{4.17}$$

if

$$\frac{1}{(\kappa-1)(n+1)/2} = \frac{\alpha}{q} + \frac{1-\alpha}{e} = \frac{\alpha}{s} + \frac{1-\alpha}{\infty}, \tag{4.18}$$

If we use that $1/e = 1/q + 1/ns$ we get $1/q = (\alpha(n+1) - 1)/ns$, or if we also use that $1/s = (n-1)(q-2)/4q$ we obtain that $\alpha = 4n/(n+1)(n-1)(q-2) + 1/(n+1)$. Hence (4.18) becomes

$$\frac{2}{(n+1)(\kappa-1)} = \frac{\alpha}{s} = \frac{1}{n+1}\left(\frac{n}{q} + \frac{1}{s}\right) = \frac{1}{2q} + \frac{n-1}{4(n+1)} = \frac{n-2\gamma}{2(n+1)}, \tag{4.19}$$

which is true by assumption. We have now concluded that (4.17) holds for some $\alpha > 0$. Now, we also have

$$\|u_m\|_{L_t^\infty L_x^e(S_T)} \leqslant M_m + \|u_0\|_{L_t^\infty L_x^e(S_T)}.$$

Hence

$$\|u_m\|_{L^{(\kappa-1)(n+1)/2}(S_T)} \leqslant 2M_0(T)^\alpha \,(2M_0 + \|u_0\|_{L_t^\infty L_x^e(S_T)})^{1-\alpha}. \quad \blacksquare$$

The rest of the proof is as in the previous case of Lemma 4.1. ∎

*Proof of existence for* $1/2 \leqslant \gamma \leqslant 3/2$ (*or equivalently* $(n+3)/(n-1) \leqslant \kappa$, $n = 2, 3$, *or* $(n+3)/(n-1) \leqslant \kappa \leqslant (n+1)/(n-3)$, $n \geqslant 4$). The key ingredient here is the following

LEMMA 4.3. *Assume that* $1/2 \leqslant \gamma \leqslant 3/2$, *where* $\gamma = n/2 - 2/(\kappa - 1)$. *Let* $q = 2(n+1)/(n-2\gamma) = (\kappa-1)(n+1)/2$ *and set*

$$M_m(T) = \| \, |D_x|^{(n-1)/2-(n+1)/q} u_m \|_{L^{2(n+1)/(n-1)}(S_T)} + \|u_m\|_{L^q(S_T)}, \tag{4.20}$$

$$N_m(T) = \|u_m - u_{m-1}\|_{L^{2(n+1)/(n-1)}(S_T \cap \Lambda_{R,0})}, \tag{4.21}$$

*where* $\Lambda_{R,0} = \{(t, x) \in \mathbb{R}^{1+n} : |x| < R - t, \ t \geqslant 0\}$ *and* $R < \infty$. *Then there is an* $\varepsilon_0 > 0$ *so that*

$$M_m(T) \leqslant 2M_0(T), \qquad N_{m+1}(T) \leqslant \tfrac{1}{2} M_m(T), \qquad \text{if} \quad M_0(T) \leqslant \varepsilon_0, \tag{4.22}$$

*in which case* (4.10) *is satisfied as well.*

*Proof of Lemma* 4.3. The proof only differs from that of Lemma 4.1 in that we have to use (3.23) instead of (3.22) in Proposition 3.5. By (3.23) applied to $\Box(u_{m+1} - u_0) = F_\kappa(u_m)$

$$M_{m+1} \leqslant C_q \, \|F'_\kappa(u_m)\|_{L^{(n+1)/2}(S_T)} M_m + M_0,$$

so we want to choose $\varepsilon_0 > 0$ in (4.22) small enough so that

$$C_q \, \|F'_\kappa(u_m)\|_{L^{(n+1)/2}(S_T)} \leqslant C \, \|u_m\|^{\kappa-1}_{L^q(S_T)} \leqslant C M_m^{\kappa-1} \leqslant C 2^{\kappa-1} M_0^{\kappa-1} \leqslant \tfrac{1}{2}.$$

Using induction we assume that the first inequality in (4.22) is true for $m = k$. Then it is true for $m = k+1$ if $C2^\kappa M_0^{\kappa-1} < 1$. The estimate for $N_m$ now follows as in the proof of Lemma 4.1, but with $s = q = 2(n+1)/(n-1)$, since we have already shown that (4.13) is true. ∎

To finish the proof of existence for this case, we first see from (3.4) that

$$\begin{aligned} M_0(T) &= \| \, |D_x|^{(n-1)/2-(n+1)/q} u_0 \|_{L^{2(n+1)/(n-1)}(S_T)} + \|u_0\|_{L^q(S_T)} \\ &\leqslant C(\|f\|_{\dot H^\gamma(\mathbb{R}^n)} + \|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}), \end{aligned} \tag{4.23}$$

where the right hand side is bounded by assumption (4.2). Since $M_0(T)$ is bounded it follows that $M_0(T_*) \leqslant \varepsilon_0$ if $T_* > 0$ is sufficiently small so that (4.22) is true. Note that if data is so small that the right side of (4.23) is $\leqslant \varepsilon_0$ then we can actually take $T_* = +\infty$.

It requires some work to prove convergence in the norm defined by (4.20). Therefore we prove convergence of the sequence $u_m$ to a function $u$ in the weaker space defined by (4.21). This together with the uniform bounds in (4.22) will imply that the limit $u$ is in the space in (4.6) defined by the norm in (4.20). Since $u_{-1} = 0$ it follows that

$$N_0(T) \leqslant C_R M_0(T),$$

for any $R>0$ so $u_m \to u$ in $L^{2(n+1)/(n-1)}_{\text{loc}}(S_T)$ and hence in $\mathscr{D}'$ and almost everywhere. Similarly, using (4.21) and Hölder's inequality, one can see that $F_\kappa(u_m)$ converges to $F_\kappa(u)$ in $L^1_{\text{loc}}$ and hence $u$ is a weak solution of (1.1).

To see that the solution satisfies the required estimates, we first notice that, by Fatou's lemma,

$$\|u\|_{L^q(S_{T_*})} \leqslant \liminf_{m \to \infty} \|u_m\|_{L^q(S_{T_*})} \leqslant 2M_0(T_*) < \infty. \tag{4.24}$$

Let us now do the argument for the other part of the norm in (4.20). We know that

$$\langle u_m, \phi \rangle \to \langle u, \phi \rangle, \qquad \text{when} \quad m \to \infty, \text{ if } \phi \in C_0^\infty.$$

But we also know that

$$\begin{aligned} |\langle u_m, \phi \rangle| \leqslant C \, \| \, |D_x|^{-\mu} \phi \|_{L^{q_2'}}, \qquad & \frac{1}{q_2'} + \frac{1}{q_2} = 1, \\ q_2 = \frac{2(n+1)}{n-1}, \qquad & \mu = \frac{n-1}{2} - \frac{n+1}{q} \end{aligned} \tag{4.25}$$

because $\| \, |D_x|^\mu u_m \|_{L^{q_2}(S_{T_*})} \leqslant C$, for all $k$. It follows directly that (4.25) holds for $u$ in place of $u_m$ if $\phi \in C_0^\infty$:

$$|\langle u, \phi \rangle| \leqslant C \, \| \, |D_x|^{-\mu} \phi \|_{L^{q_2'}}.$$

This means that $\| \, |D_x|^\mu u \|_{L^{q_2}} \leqslant C$.

This proves that $u$ satisfies (4.5) and (4.6). (4.4) then follows from (3.20) in Corollary 3.4 with $F = F_\kappa(u)$. In fact by (3.25), (4.5) and (4.6) implies that

$$\| \, |D_x|^{\gamma - 1/2} F_\kappa(u) \|_{L^{2(n+1)/(n+3)}(S_T)} \leqslant C, \tag{4.26}$$

for some constant $C < \infty$. This finishes the proof of existence for $1/2 \leqslant \gamma \leqslant 3/2$. ∎

*Proof of existence for* $n \geqslant 4$, $\kappa \in \mathbb{Z} \cap ((n+1)/(n-3), \infty)$, *or, equivalently,* $\gamma > 3/2$. For technical reasons explained before, for this range of $\kappa$, we are assuming that $F_\kappa(u) = \pm u^\kappa$. If the $u_m$ are defined as before, arguing as above, we just need to prove the following

LEMMA 4.4. *Let now*

$$M_m(T) = \sup_{2(n+1)/(n-1) \leqslant q \leqslant (\kappa-1)(n+1)/2} \| \, |D_x|^{(n+1)/q - 2/(\kappa-1)} u_m \|_{L^q(S_T)},$$

*and let* $N_m(T)$ *be as in* (4.21). *Then there is an* $\varepsilon_0 > 0$ *so that*

$$M_m(T) \leqslant 2M_0(T), \qquad N_{m+1}(T) \leqslant \tfrac{1}{2} N_m(T), \qquad \text{if } \; M_0(T) \leqslant \varepsilon_0.$$

*Proof.* We shall only prove the estimate for $M_m$ since the one for $N_m$ follows from the proof of Lemma 4.3.

If we apply (3.8) to $\Box(u_{m+1} - u_0) = \pm u_m^\kappa$, we get

$$M_{m+1} \leqslant M_0 + C_\kappa \, \| \, |D_x|^{(n-1)/2 - 2/(\kappa-1)} \, u_m^\kappa \|_{L^{2(n+1)/(n+3)}(S_T)}. \tag{4.27}$$

To control the last term, we need to use the following variation on (3.25) (see [3], [6], [17]):

$$\| \, |D_x|^\sigma (fg) \|_{L^p} \leqslant C \, \| \, |D_x|^\sigma f \|_{L^{r_2}} \, \|g\|_{L^{r_1}} + C \, \|f\|_{L^{s_1}} \, \| \, |D_x|^\sigma g \|_{L^{s_2}},$$
$$C = C_{r_j, s_j, p, \sigma},$$

if $0 \leqslant \sigma \leqslant 1$, $1 < r_j$, $s_j < \infty$, and $1/p = 1/r_1 + 1/r_2 = 1/s_1 + 1/s_2$. If we use this and the fact that for a given multi-index $\alpha$ and $1 < p < \infty$

$$\|D^\alpha f\|_{L^p} \leqslant C_{p, \alpha} \, \| \, |D_x|^{|\alpha|} f \|_{L^p},$$

we find that the last term in (4.27) can be dominated by a finite sum of terms of the form

$$\prod_{j=1}^{\kappa} \| \, |D_x|^{\alpha_j} u_m \|_{L^{q_j}(S_T)}, \tag{4.28}$$

with $0 \leqslant \alpha_j \leqslant (n-1)/2 - 2/(\kappa - 1)$ being fixed numbers satisfying

$$\sum_{j=1}^{\kappa} \alpha_j = \frac{n-1}{2} - \frac{2}{\kappa - 1},$$

and $2(n+1)/(n-1) \leqslant q_j < \infty$ being arbitrary numbers satisfying

$$\sum_{j=1}^{\kappa} \frac{1}{q_j} = \frac{n+3}{2(n+1)}. \tag{4.29}$$

But if we let $q_j$ be determined by

$$\frac{n+1}{q_j} - \frac{2}{\kappa - 1} = \alpha_j, \tag{4.30}$$

then $2(n+1)/(n-1) \leqslant q_j \leqslant (\kappa - 1)/(n+1)/2$, and

$$\sum_{j=1}^{\kappa} \frac{1}{q_j} = \frac{1}{(n+1)} \sum_{j=1}^{\kappa} \left( \alpha_j + \frac{2}{\kappa - 1} \right) = \frac{n+3}{2(n+1)},$$

so that (4.29) is satisfied. Also, since (4.30) holds, we can dominate each factor in (4.28) by $M_m(T)$, and, hence, by (4.27),

$$M_{m+1} \leqslant M_0 + C_\kappa M_m^\kappa,$$

which as before gives us the desired bound for $M_{m+1}$ if $M_0(T)$ is small enough. ∎

*Proof of global existence for small norms.* The global existence proof for small norms as in Theorem 2.2 is the same as the local existence proof. If (2.3) is satisfied for $\varepsilon > 0$ so small that the right hand side of (4.23) is $\leqslant \varepsilon_0$ then (4.22) is satisfied for any $T$, in particular for $T = \infty$. Therefore $u_m$ converges in the the norms defined by (4.20) and (4.21) with $T = \infty$. Notice that the proof of Lemmas 4.3 and 4.4, see (4.26), shows that the global solution to (1.1) in Theorem 2.2 satisfies

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} \left| |D_x|^{\gamma(\kappa) - 1/2} F_\kappa(u)\right|^{2(n+1)/(n+3)} dx\, dt < C. \tag{4.31}$$

This will be the key ingredient in the proof of the scattering part of Theorem 2.2.

*Proof of existence for $0 \leqslant \gamma < 1/2 - 1/(n-1)$, or, equivalently, $(n+3)/n \leqslant \kappa < (n+1)^2/((n-1)^2 + 4)$.*

We need to show that if $T_* > 0$ is small there is a weak solution of (4.3) in $S_{T_*}$ satisfying (4.4) as well as

$$u \in L_t^s L_x^q(S_{T_*}), \qquad q = \frac{2(n+1)}{n+1-4\gamma}, \qquad s = \frac{4q}{(n-1)(q-2)}.$$

To do this we let $u_m$ be defined as in (4.7) and (4.8). Then, as before, we need to show that there is a $T_* > 0$ so that the $u_m$ converge to $u$ as in (4.9), with $u$ satisfying the above conditions.

If we repeat the arguments at the beginning of this section, we see that this follows from Corollary 3.4 and the following

LEMMA 4.5. *Assume that $n \geqslant 4$ and that $0 \leqslant \gamma < 1/2 - 1/(n-1)$, for $\gamma = \gamma(\kappa)$ as in (2.5). Also, if $q = q(\kappa)$ and $s = s(\kappa)$ are as in (2.6), set*

$$M_m(T) = \|u_m\|_{L_t^s L_x^q(S_T)}, \qquad N_m(T) = \|u_m - u_{m-1}\|_{L_t^s L_x^q(S_T)}.$$

*Then there is an $\varepsilon_0 > 0$ so that*

$$M_m(T) \leqslant 2M_0(T),$$
$$N_{m+1}(T) \leqslant \tfrac{1}{2} N_m(T), \qquad \text{if} \quad 2M_0(T)\, T^{(1/2-\kappa/s)/(\kappa-1)} \leqslant \varepsilon_0.$$

*Proof of Lemma* 4.4. Let $p=q/\kappa$. Then $p$ and $q$ satisfy the relation in (3.15). Therefore if we use (3.14) we get

$$\begin{aligned}\|u_{m+1}-u_{k+1}\|_{L_t^s L_x^q(S_T)} &\leqslant C_q \|F_\kappa(u_m)-F_\kappa(u_k)\|_{L_t^2 L_x^p(S_T)}\\ &\leqslant C_q \|V_\kappa(u_m, u_k)\|_{L_t^{\bar{s}} L_x^{\bar{q}}(S_T)} \|u_m-u_k\|_{L_t^s L_x^q(S_T)},\end{aligned}$$

if $1/\bar{s}=1/2-1/s$, $1/\bar{q}=1/p-1/q$, and, as before, $V_\kappa(u, v)=(F_\kappa(u)-F_\kappa(v))/(u-v)$. A calculation shows that $(\kappa-1)\bar{q}=q$ and that $s>(\kappa-1)\bar{s}$. Thus, since $|V_\kappa(u, v)|\leqslant C(|u|^{\kappa-1}+|v|^{\kappa-1})$, we can use Hölder's inequality to get

$$\|V_\kappa(u_m, u_k)\|_{L_t^{\bar{s}} L_x^{\bar{q}}(S_T)} \leqslant CT^{1/2-\kappa/s}(\|u_m\|^{\kappa-1}_{L_t^s L_x^q(S_T)}+\|u_k\|^{\kappa-1}_{L_t^s L_x^q(S_T)}).$$

Thus,

$$\|u_{m+1}-u_{k+1}\|_{L_t^s L_x^q(S_T)} \leqslant \tfrac{1}{2} \|u_m-u_k\|_{L_t^s L_x^q(S_T)}).$$

if $\varepsilon_0>0$ is sufficiently small and $M_j(T)\, T^{(1/2-\kappa/s)/(\kappa-1)}\leqslant \varepsilon_0$. To prove the first part of the lemma, as in the proof of Lemma 4.1, one uses induction and applies this inequality with $k=-1$. The other part of the lemma comes from taking $k=m-1$. ∎

## 5. Propagation of Regularity and Uniqueness

So far, we have mostly been concerned with finding minimal regularity assumptions on Cauchy data that imply existence of a local solution. Another interesting problem is to determine how much regularity is needed to continue a smooth solution, that is, which norms are forced to blow up when a locally smooth solution ceases to be smooth. For the equation

$$\Box u=u^2, \qquad u(0, x)=f(x), \qquad u_t(0, x)=g(x),$$

in $\mathbb{R}^{1+3}$ it was shown in [21] that one can find data $(f, g)\in \dot{H}^0(\mathbb{R}^3)\times \dot{H}^{-1}(\mathbb{R}^3)$ for which there is no local distributional solution with $\|u\|_{L^2([0,T]\times\mathbb{R}^3)}$ bounded for any $T>0$. Still, it follows from Theorem 5.1 below that for a solution with data $(f, g)\in \dot{H}^\gamma(\mathbb{R}^3)\times \dot{H}^{\gamma-1}(\mathbb{R}^3)$, for some $\gamma>0$, $\|u(T,\cdot)\|_{\dot{H}^\gamma}$ remains bounded as long as $\|u\|_{L^2([0,T]\times\mathbb{R}^3)}$ is bounded. There is a very simple proof by Levin (see Reed and Simon [27, p. 311]), showing that the above equation cannot have a global smooth solution for a very large class of data. They argued by contradiction, showing that, for such fixed data, there is a $T<\infty$ such that if we have a smooth solution for $t<T$ then $\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)}\to\infty$ as $t\to T$. It now follows from our result

that the solution actually is smooth when $t<T$ so indeed the $L^2$ norm blows up. In general we show below that smoothness for the equation (1.1) propagates as long as

$$\int_0^T \int |u(t,\ )|^{((n+1)/2)(\kappa-1)}\,dx\,dt<\infty,$$

which also is what is needed for uniqueness. We also show that the integral of the above quantity over a backward light cone has to become infinite when the solution blows up. In fact, in any backward light cone with the top on the boundary of the existence space, see Definition 5.5 and Theorem 5.6.

THEOREM 5.1. *Assume that* $V\in L^{(n+1)/2}([0,T]\times\mathbb{R}^3)$ *and that* $(f,g)\in \dot H^{\gamma}(\mathbb{R}^n)\times\dot H^{\gamma-1}(\mathbb{R}^n)$, *where* $|\gamma-1/2|<1/(n-1)$ *if* $n\geqslant 3$ *and* $|\gamma-1/2|<1/4$ *if* $n=2$. *Then the equation*

$$\Box u=Vu,\qquad u(0,x)=f(x),\qquad u_t(0,x)=g(x) \tag{5.1}$$

*has a unique solution in*

$$(u,\partial_t u)\in C_b([0,T];\dot H^{\gamma}(\mathbb{R}^n)\times\dot H^{\gamma-1}(\mathbb{R}^n))\quad \text{and}\quad u\in L^s_tL^q_x([0,T]\times\mathbb{R}^n),$$

*where* $q=2(n+1)/(n+1-4\gamma)$, *and* $s=4q/(n-1)(q-2)$. *Moreover, this solution satisfies the estimate*

$$\|u(T,\cdot)\|_{\dot\gamma}\leqslant 2\exp\left(K_\gamma\int_0^T\int |V(t,x)|^{(n+1)/2}\,dx\,dt\right)\|u(0,\cdot)\|_{\dot\gamma}, \tag{5.2}$$

*for some universal constant* $K_\gamma$. *On the other hand, assume that* $u\in L^q([0,T]\times\mathbb{R}^n)$ *and* $|D_x|^{\gamma-1/2}u\in L^{2(n+1)/(n-1)}([0,T]\times\mathbb{R}^n)$, *where* $1/2\leqslant\gamma=n/2-(n+1)/q\leqslant 3/2$, *satisfies*

$$\Box u=F(u).$$

*Then*

$$\|u(T,\cdot)\|_{\dot\gamma}\leqslant 2\exp\left(K_\gamma\int_0^T\int |F'(u(t,x))|^{(n+1)/2}\,dx\,dt\right)\|u(0,\cdot)\|_{\dot\gamma}. \tag{5.3}$$

*Remark.* The condition that $u\in L^s_tL^q_x([0,T]\times\mathbb{R}^n)$ is needed to ensure uniqueness and the estimate (5.2). There is, however, always a solution that satisfies this condition, and, in a sense, this is the natural solution. In fact, if initial data are smooth the solution we construct satisfies (5.2). Since the equation is linear (5.2) shows that the solution operator extends to a continuous operator in the space defined by the norms in (5.2). In the space defined by the image of the solution operator, we of course, by (5.2), have unqiueness. The uniqueness problem only comes up when we consider general distributional solutions which anyway might be too weak of a concept of solutions.

*Proof of Theorem* 5.1. Let us first prove (5.1). We want to use Proposition 3.5 applied to the equation $\Box(u-u_0)=Vu$ where $u_0$ is the solution of $\Box u_0=0$ with the same data as $u$ when $t=0$. Let $T_1$ be the largest number such that

$$\|V\|_{L^{(n+1)/2}([0,\,T]\times\mathbb{R}^n)}\leqslant\varepsilon_\gamma,\qquad \text{for}\quad T\leqslant T_1, \tag{5.4}$$

where $\varepsilon_\gamma$ is a universal constant to be determined. In particular, if $\varepsilon_\gamma\leqslant C_q/2$, then by (3.22)

$$\|u-u_0\|_{L_t^sL_x^q([0,\,T]\times\mathbb{R}^n)}\leqslant\tfrac{1}{2}\|u\|_{L_t^sL_x^q([0,\,T]\times\mathbb{R}^n)},$$

which implies that

$$\|u\|_{L_t^sL_x^q([0,\,T]\times\mathbb{R}^n)}\leqslant 2\,\|u_0\|_{L_t^sL_x^q([0,\,T]\times\mathbb{R}^n)},\qquad \text{for}\quad T\leqslant T_1, \tag{5.5}$$

provided that the left side is bounded. This immediately implies uniqueness of solutions $u_1,\ u_2\in L_t^sL_x^q([0,\,T]\times\mathbb{R}^n)$, because $u=u_1-u_2$ is a solution of the same equation with the corresponding $u_0=0$. As in Section 4 we can construct a solution of (5.1) for $t\leqslant T_1$ such that the left side of (5.5) is bounded if $q=2(n+1)/(n+1-4\gamma)$, and $s=4q/(n-1)(q-2)$, by putting $u=\lim_{k\to\infty}u_k$ where $u_{-1}=0$ and $\Box u_{k+1}=Vu_k$ with data $u_k(0,x)=f(x)$, $\partial_tu_k(0,x)=g(x)$. By (3.5) the right side of (5.5) is bounded by a constant times $\|u(0,\cdot)\|_{\dot\gamma}$ if $\gamma=(1/2-1/q)(n+1)/2$. On the other hand, by (3.12), with $(n+5)/4-(n+1)/2p=\gamma$, (i.e., $1/p-1/q=2/(n+1)$ as before), and (3.24)

$$\|(u-u_0)(T,\cdot)\|_{\dot\gamma}\leqslant C\,\|V\|_{L^{(n+1)/2}([0,\,T]\times\mathbb{R}^n)}\,\|u\|_{L_t^sL_x^q([0,\,T]\times\mathbb{R}^n)}.$$

Hence

$$\begin{aligned}\|(u-u_0)(T,\cdot)\|_{\dot\gamma}&\leqslant C\,\|V\|_{L^{(n+1)/2}([0,\,T]\times\mathbb{R}^n)}\,\|u(0,\cdot)\|_{\dot\gamma}\\&\leqslant\tfrac{1}{2}\|u(0,\cdot)\|_{\dot\gamma},\qquad \text{for}\quad T\leqslant T_1,\end{aligned} \tag{5.6}$$

if (5.4) is satisfied with $C\varepsilon_\gamma\leqslant 1/2$. Since, by the energy identity, $\|u_0(T,\cdot)\|_{\dot\gamma}=\|u(0,\cdot)\|_{\dot\gamma}$ we finally conclude that

$$\|u(T,\cdot)\|_{\dot\gamma}\leqslant 2\,\|u(0,\cdot)\|_{\dot\gamma},\qquad \text{if}\quad T\leqslant T_1, \tag{5.7}$$

provided that (5.4) is satisfied. This proves (5.2) when (5.4) holds. If $T$ is so large that (5.4) is not true we just repeat this argument: For a given $T$ choose $0 = T_0 < T_1 < T_2 < T_3 \cdots < T_N < T_{N+1} = T$ such that

$$\|V\|_{L^{(n+1)/2}([T_k, T_{k+1}] \times \mathbb{R}^n)} = \varepsilon_\gamma, \qquad k = 0, ..., N-1.$$

Then

$$\int_0^T \int |V(t, x)|^{(n+1)/2} \, dx \, dt = \sum_{k=0}^{N} \|V\|^{(n+1)/2}_{L^{(n+1)/2}([T_k, T_{k+1}] \times \mathbb{R}^n)} \geqslant N\varepsilon_\gamma^{(n+1)/2}. \tag{5.8}$$

Repeating the argument that lead to (5.7) $N$ times gives

$$\|u(T_N, \cdot)\|_\gamma \leqslant 2^N \|u(0, \cdot)\|_\gamma,$$

or if we use (5.8)

$$\|u(T, \cdot)\|_\gamma \leqslant 2 \exp\left(\log(2)\, \varepsilon_\gamma^{\;(n+1)/2} \int_0^T \int |V(t, x)|^{(n+1)/2} \, dx \, dt\right) \|u(0, \cdot)\|_\gamma.$$

The proof of (5.3) is similar; one uses (3.4), (3.11), (3.25) and Lemma 4.3 instead of (3.5), (3.12), (3.24) and Lemma 4.1. ∎

THEOREM 5.2. *Assume that we have two solutions $u_1$ and $u_2$ of*

$$\Box u_i = F_\kappa(u_i), \qquad \text{and} \qquad u_i(0, x) = f(x), \qquad \partial_t u_i(0, x) = g(x),$$

*where $F_\kappa$ is as in* (1.2). *Assume also that*

$$\int_0^T \int |u_i(t, x)|^{(\kappa-1)(n+1)/2} \, dx \, dt \leqslant C, \qquad i = 1, 2. \tag{5.9}$$

*Then $u_1 = u_2$ for $t < T$ if*

$$\kappa > 1 + \frac{4n}{(n-1)(n+1)}.$$

*The same conclusion is also true if we in addition assume that*

$$u_i \in L_t^s L_x^q([0, T] \times \mathbb{R}^n), \qquad s = \frac{4q}{(n-1)(q-2)}, \tag{5.10}$$

*for some $q$ as in* (3.21). *In particular, if $\kappa_0 < \kappa \leqslant 1 + 4n/(n-1)(n+1)$, where $\kappa_0$ is as in Theorem* 2.1, *we can take $q = (\kappa - 1)(n+1)/2$.*

*Proof.* The difference $u = u_1 - u_2$ satisfies the equation

$$\Box u = Vu, \qquad u_i(0, x) = \partial_t u_i(0, x) = 0, \qquad \text{where} \quad V = \frac{F_\kappa(u_1) - F_\kappa(u_2)}{u_1 - u_2}.$$

Since by assumption $|F'_\kappa(u)| \leqslant C\,|u|^{\kappa-1}$ it follows that

$$\int_0^T \int |V(t, x)|^{(n+1)/2}\, dx\, dt \leqslant C'. \tag{5.11}$$

Our uniqueness results for the nonlinear equation follow from uniqueness for the linear equation with the assumption (5.11), essentially like in the proof of Theorem 5.1. It follows from (3.22) that

$$\|u\|_{L_t^s L_x^q(S_T \cap \Lambda_{R,0})} \leqslant C_q\, \|V\|_{L^{(n+1)/2}([0,T] \times \mathbb{R}^n)}\, \|u\|_{L_t^s L_x^q(S_T \cap \Lambda_{R,0})}, \tag{5.12}$$

if $s = 4q/(n-1)(q-2)$ and $q$ is as in (3.21) and also by Harmse's inequality [10], (8.2), if $s = q$ and $2n/(n-1) < q \leqslant 2(n+1)/(n-1)$. Here $\Lambda_{R,0} = \{(t, x) : |x| + t \leqslant R\}$ is any backward light cone. If we choose $T$ sufficiently small it follows from (5.11) and (5.12) that $\|u\|_{L_t^s L_x^q(S_T \cap \Lambda_{R,0})} = 0$, provided that this norm is bounded. Then repeating the argument a finite number of times would yield that $u(t, x) = 0$ for $t < T$. Our first claim is that if $\kappa > 1 + 4n/(n-1)(n+1)$ then (5.9) alone implies uniqueness. First if $\kappa \geqslant (n+3)/(n-1)$ then, by (5.9),

$$\|u\|_{L^{2(n+1)/(n-1)}(S_T \cap \Lambda_{R,0})} \leqslant C_R\, \|u\|_{L^{(\kappa-1)(n+1)/2}(S_T \cap \Lambda_{R,0})} < \infty,$$

so the result follows in this case. On the other hand if $1 + 4n/(n-1)(n+1) < \kappa \leqslant (n+3)/(n-1)$ then $q = (\kappa-1)(n+1)/2 > 2n/(n-1)$ so again the result follows. In the second range of $\kappa$ the proof follows in the same way from (5.10) and (5.12). ∎

*Proof of uniqueness for the solution in Theorem* 2.1 *and Theorem* 2.3. If $n \geqslant 6$ then $1 + 4n/(n-1)(n+1) < \kappa_0 = (n+1)^2/((n-1)^2 + 4)$ so the uniqueness in this case follows directly from the first part of Theorem 5.2. Also $(n+3)/(n-1) > 1 + 4n/(n-1)(n+1)$ so the uniqueness follows for general $n \geqslant 2$ if $\kappa \geqslant (n+3)/(n-1)$. On the other hand, we have an additional assumption, apart from (5.9) that we have not used yet, namely, that

$$u_i \in C_b([0, T];\, \dot H^\gamma(\mathbb{R}^n) \times \dot H^{\gamma-1}(\mathbb{R}^n)), \qquad \text{where } \gamma = \frac{n+1}{4} - \frac{2}{\kappa-1} > \frac{n-3}{2(n-1)}.$$

This implies, by Sobolev's, lemma that $u_i \in L_t^\infty L_x^{2/(1-\gamma/n)}$. If $n = 3$ this directly implies that $u$ is in the space (5.10) for some $q > 2$ which gives uniqueness if $\kappa > 2$. On the other hand if $n = 2, 4, 5$ the best we can do is to assume that in addition $u_i$ satisfies (5.10) with $q = (\kappa - 1)(n+1)/2$ which is true for the local solution we constructed in Section 4.

The proof of uniqueness in Theorem 2.3 follows in a similar way from the proof of Lemma 4.5. ∎

Next we will present some results on which norms have to blow up when the solution cannot be extended any further. Let us first give some definitions.

DEFINITION 5.3. Assume that $\kappa$, $f$ and $g$ satisfy the conditions for local existence and uniqueness in Theorem 2.1. Then the lifespan, $T(f, g)$ (or $T(u)$), is the supremum over $T > 0$ for which (1.1) has a solution $u \in L_t^{s(\kappa)} L_x^{q(\kappa)}([0, T] \times \mathbb{R}^n)$, where

$$q(\kappa) = \frac{n+1}{2} (\kappa - 1)$$

and

$$s(\kappa) = \begin{cases} \dfrac{4q(\kappa)}{(n-1)(q(\kappa)-2)} & \text{if } \kappa_0 < \kappa < \dfrac{n+3}{n-1}, \\ q(\kappa) & \text{if } \dfrac{n+3}{n-1} \leqslant \kappa. \end{cases} \tag{5.13}$$

DEFINITION 5.4. By an influence domain we mean an open set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ such that $\{0\} \times \mathbb{R}^n \subset \Omega$ and $(t_0, x_0) \in \Omega$ implies that the backward light cone $\Lambda_{t_0, x_0} \subset \Omega$. Here

$$\Lambda_{t_0, x_0} = \begin{cases} \{(t, x) : |x - x_0| \leqslant t_0 - t,\ t \geqslant 0\}, & \text{if } t_0 > 0, \\ \{(t, x) : |x - x_0| \leqslant t - t_0,\ t \leqslant 0\}, & \text{if } t_0 < 0. \end{cases}$$

The next one generalizes Definition 5.3.

DEFINITION 5.5. The domain of existence $\Omega(f, g)$ (or $\Omega(u)$), is the union of all influence domains $\Omega_i$ for which (1.1) has a solution $u_i$ in $\Omega_i$ with $u_i \in L_t^{s(\kappa)} L_x^{q(\kappa)}(\Lambda_{t_0, x_0})$ for all cones $\Lambda_{t_0, x_0} \subset \Omega_i$.

*Remark.* It follows directly from the above definition that $\Omega(f, g)$ is an influence domain and that there is a solution $u$ of (1.1) in $\Omega(f, g)$ with the above property. In fact, if $u_i$ is a solution in $\Omega_i$, $i = 1, \ldots$, then we can simply define our solution $u$ to be equal to $u_i$ in $\Omega_i$, (see Theorem 3.6 in [20]), because in the intersection of two such influence domains we have uniqueness by the proof of Theorem 5.2. Notice also that if $T(f, g)$ is the lifespan, we have $T(f, g) = \inf\{t > 0 : (t, x) \in \partial\Omega(f, g)\}$.

We now turn to a sequence of results about the domain of existence and the blow-up behavior of solution to (1.1).

THEOREM 5.6. *Assume that* $\kappa_0 < \kappa \leqslant (n+1)/(n-3)$ *and let* $T = T(u)$ *be the lifespan and* $\Omega = \Omega(u)$ *be the domain of existence of a solution* $u$ *to* (1.1). *Then* $\|u\|_{L^q([0,T]\times\mathbb{R}^n)} = \infty$. *Moreover, if* $\kappa_0 < \kappa \leqslant (n+3)/(n-1)$ *and* $(t_0, x_0) \in \partial\Omega$ *then* $\|u\|_{L^q(A_{t_0,x_0})} = \infty$.

*Proof.* If $\|u\|_{L^q([0,T]\times\mathbb{R}^n)} < \infty$ then it would follow from Theorem 5.1 that $\|u(T,\cdot)\|_{\dot{\gamma}} < \infty$ so by Theorem 2.1 the solution can be extended for some further time, contradicting the maximality of $T$. To prove the second part we also want to argue by contradiction. We want to show that if $\|u\|_{L^q(A_{t_0,x_0})} \leqslant C$ then there exist a larger cone $A' \supset A_{t_0,x_0}$ and a solution $w \in L_t^s L_x^q(A')$. Such an extension is obtained if we can solve the corresponding integral equation for $v$ (see the proof of Theorem 3.6 in [20])

$$v = E * ((1-\chi_{A_{t_0,x_0}})\, F_\kappa(v)) + v_0,$$

where (5.14)

$$v_0 = (1-\chi_{A_{t_0,x_0}})(E * (\chi_{A_{t_0,x_0}} F_\kappa(u)) + u_0),$$

in $A'$. Here $\chi_{A_{t_0,x_0}}$ is the characteristic function of $A_{t_0,x_0}$, $E$ is the forward fundamental solution of $\Box$ and $u_0$ is the solution of $\Box u_0 = 0$ with the same data as $u$ when $t=0$. In fact, then $w = (1-\chi_{A_{t_0,x_0}})v + \chi_{A_{t_0,x_0}} u$ defines a solution of $w = E * (F_\kappa(w)) + u_0$ in $A'$, contradicting the maximality of $\Omega$. First, it follows as in the proof of Theorem 5.1 that $\|u\|_{L^q(A_{t_0,x_0})} \leqslant C$ implies that $\|u\|_{L_t^s L_x^q(A_{t_0,x_0})} \leqslant C'$ provided that $\|u_0\|_{L_t^s L_x^q(A_{t_0,x_0})} \leqslant C''$. It then follows, as in the proof of existence in Section 4 and the proof of Theorem 5.1, first that $\|v_0\|_{L_t^s L_x^q(A')} \leqslant C''$, and then that we can obtain a solution $v$ of (5.14) by iteration in $A'$ if it is so small that $\|v_0\|_{L_t^s L_x^q(A')} = \|v_0\|_{L_t^s L_x^q(A' \setminus A_{t_0,x_0})} \leqslant \varepsilon_0$. ∎

Theorem 5.7. *Let* $n=3$ *and* $2 < \kappa < 3$. *Assume that* $u$ *is the solution of* (2.1) *with* $|F_\kappa(u)| \leqslant C|u|^\kappa$, $|F'_\kappa(u)| \leqslant C|u|^{\kappa-1}$, and $|F''_\kappa(u)| \leqslant C|u|^{\kappa-2}$. *If initial data* $f \in C_0^3$ *and* $g \in C_0^2$ *then* $u \in C^2(\Omega(u))$.

*Proof.* From the standard existence theorem for $C^2$ (see e.g. [20, Theorems 3.6–3.7]), we know that it suffices to prove an $L^\infty$ bound for $u$ in any cone $A \subset \Omega(u)$. Choose another cone $A' \subset \Omega$, such that $A$ is strictly contained in the interior of $A'$. As in the proof of Theorem 5.1, one can use Strichartz's inequality to first conclude that $\|\partial^\alpha u\|_{L^4(A')} \leqslant C$, for $|\alpha| \leqslant 1$ and then that $\|\partial^\alpha u\|_{L^4(A')} \leqslant C'$, for $|\alpha| = 2$. This implies that $u \in L^\infty(A)$. ∎

## 6. Sharpness

Assume that $\kappa$ is as in Theorem 2.1, that is $\kappa_0 < \kappa$, where $\kappa_0 = (n+1)^2/((n-1)^2+4)$ if $n \geqslant 3$ and $\kappa_0 = 3$ if $n = 2$. In this section we shall show that the special case of (1.1),

$$\begin{cases} \Box u = u_+^\kappa \\ u(0, x) = f \in \dot{H}^\gamma(\mathbb{R}^n), \qquad \partial_t u(0, x) = g \in \dot{H}^{\gamma-1}(\mathbb{R}^n) \end{cases} \tag{6.1}$$

is ill-posed if $u_+^\kappa = u^\kappa$ when $u \geqslant 0$ and 0 otherwise, and

$$\gamma < \gamma(\kappa) = \begin{cases} \dfrac{n+1}{4} - \dfrac{1}{\kappa - 1}, & \kappa_0 < \kappa \leqslant \dfrac{n+3}{n-1}, \\ \dfrac{n}{2} - \dfrac{2}{\kappa - 1}, & \text{if } \kappa \geqslant \dfrac{n+3}{n-1}. \end{cases}$$

More precisely, we shall construct a sequence of $C_0^\infty$ data, whose $\|\cdot\|_\gamma$ norm tends to 0, while, at the same time, the lifespan of the corresponding solutions also tends to 0. In the special case $n = 3$, where the fundamental solution is positive, one can use this construction to also produce data for which there is no local solution (see Lindblad [21]).

The method we shall use is simple. One starts with an initially smooth radial solution of (6.1) that blows up for some finite time $T$. One can construct such a function by using a solution of the ordinary differential equation $f''(t) = (f(t))_+^\kappa$ that blows up when $t = T$ if one cuts off the corresponding data when $|x| \geqslant T$. Then in the easy superconformal case where $\kappa \geqslant (n+3)/(n-1)$ one gets the ill-posedness simply using the scaling argument described in the introduction. In the harder subconformal case $\kappa < (n+3)/(n-1)$ one gets the ill-posedness by studying how the domain of existence and the norms transform under a certain combination of a Lorentz transformation and a scaling.

Recall Definition 5.3 for the lifespan $T(f, g)$ and Definition 5.5 for the domain of existence $\Omega(f, g)$ of a solution $u$ to (1.1). In order to state a more general result about the lifespan of a possibly weaker solution which is not in $L_t^s L_x^q([0, T] \times \mathbb{R}^n)$ but in $C_b([0, T]; \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n))$, we need a new definition.

Definition 6.1. Let the weak lifespan $T^\gamma(f, g)$ be the supremum over all $T$ such that there is a solution $w$ of (1.1), satisfying $(w, \partial_t w) \in C_b([0, T]; \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n))$ and $w = u$ in $\Omega(f, g) \cap \{(t, x) : 0 \leqslant t < T\}$ (see Definition 5.5).

*Remark.* Notice that if $|\gamma - 1/2| < 1/(n-1)$ then it follows from the proof of Theorem 5.1 that $T^\gamma(f, g) \geqslant T(f, g)$.

Our main ill-posedness results are:

THEOREM 6.2. *Let $\kappa > \kappa_0$ and assume that $n \geqslant 2$. Then there is $\varepsilon > 0$ such that if $\gamma - \varepsilon < \gamma < \gamma(\kappa)$ the following is true. There is a sequence of data $(f_j, g_j)$, which are smooth and supported in the unit ball, for which the corresponding solutions $u_j$ of (6.1) satisfy,*

$$T(f_j, g_j) \to 0, \qquad T^{\gamma}(f_j, g_j) \to 0, \qquad as \quad j \to \infty \tag{6.2}$$

*(see Definitions 5.3 and 6.1) and*

$$\int \left| |D_x|^{\gamma} f_j(x) \right|^2 + \left| |D_x|^{\gamma - 1} g_j(x) \right|^2 dx \to 0. \tag{6.3}$$

*Remark.* The condition in Definition 6.1 that $w_j = u_j$ for $(t, x) \in \Omega(f_j, g_j)$ is natural. The theorem says that the problem is ill-posed in $\dot{H}^{\gamma}(\mathbb{R}^n) \times \dot{H}^{\gamma-1}(\mathbb{R}^n)$ if $\gamma < \gamma(\kappa)$, in the sense that either there is no solution in this space, or else there is no uniqueness. It is also worth noting that we do not actually claim that $\|w_j(t, \cdot)\|_{\dot{\gamma}} \to \infty$ when $t \to T^{\gamma}(w_j)$, but only that there cannot be a solution $w_j$ with this norm bounded for $0 \leqslant t \leqslant T^{\gamma}(w_j)$.

THEOREM 6.3. *Let $\kappa > \kappa_0$ and assume that $n \geqslant 2$ and $\kappa \geqslant (n+3)/(n-1)$, or $n = 3$ and $2 < \kappa < 3$. There is $\varepsilon > 0$ such that if $\gamma - \varepsilon < \gamma < \gamma(\kappa)$ the following is true. Then there is a sequence of data $(f_j, g_j)$, which are smooth and supported in the unit ball, for which the corresponding solutions $u_j$ of (6.1) satisfy,*

$$T(f_j, g_j) \to 0$$

*and* (6.4)

$$\int \left| |D_x|^{\gamma} f_j(x) \right|^2 + \left| |D_x|^{\gamma - 1} g_j(x) \right|^2 dx \to 0, \qquad as \quad j \to \infty.$$

*Moreover for each of these $u_j$ we have*

$$\|u_j(t, \cdot)\|_{\dot{\gamma}} \to \infty, \qquad as \quad t \to T(f_j, g_j), \tag{6.5}$$

*hence showing that $T^{\gamma}(f, g) = T(f, g)$.*

The proof of Theorem 6.2 is a direct consequence of the following two lemmas. Figure 6.1 below illustrates the blow-up of $u_1$ in Lemma 6.4 and the cut-off of initial data in Lemma 6.5.

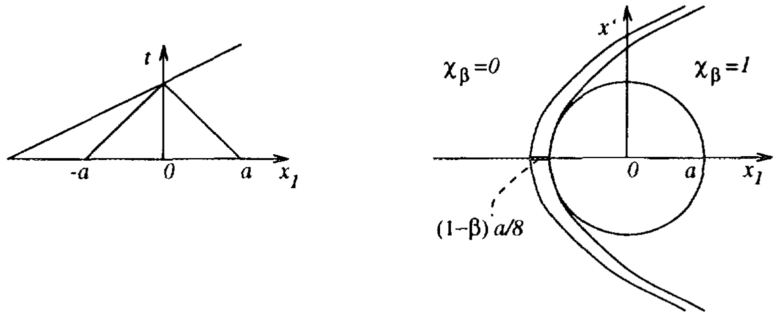



FIG. 6.1. The domain of existence of $u_1$ and the cut-off of initial data.

LEMMA 6.4. *If $0 \leqslant \beta < 1$ then*

$$u_1(t, x) = \frac{c_\alpha (1-\beta^2)^{\alpha/2}}{(a-(t-\beta x_1))^\alpha}, \qquad \alpha = \frac{2}{\kappa - 1}, \qquad c_\alpha = (\alpha(\alpha+1))^{\alpha/2}. \tag{6.6}$$

*satisfies* (6.1) *when $t - \beta x_1 < a$. Moreover, if $f, g \in C^\infty$ satisfies $f(x) = u_1(0, x)$ and $g(x) = \partial_t u_1(0, x)$, when $|x| \leqslant a + \varepsilon$, for some $\varepsilon > 0$, then the lifespan for the solution of* (6.1) *satisfies $T(f, g) \leqslant a$. The weak lifespan also satisfies $T^\gamma(f, g) \leqslant a$, if $\gamma > n/2 - 2/(\kappa-1)$ or if $\gamma \geqslant 0$ and $\beta = 0$.*

*Proof of Lemma* 6.4. After scaling and applying a Lorentz transformation to the ode solution $c_\alpha/(1-t)^\alpha$ we get the solutions (6.6). If $T(f, g) > a$ then the proof of uniqueness in Theorem 5.2 gives that $u(t, x) = u_1(t, x)$ when $|x| + |t| < a + \varepsilon$ and $t < a$. But if follows from the homogeneity of $u_1$ that

$$\int_0^a \int_{|x| \leqslant a - t} |u_1(t, x)|^{(\kappa-1)(n+1)/2} \, dx \, dt = \infty,$$

which contradicts the fact that that $\|u\|_{L_t^r L_x^q([0, a] \times \mathbb{R}^n)} < \infty$.

Let us now prove the last part of the lemma. It also follows directly from the homogeneity of $u_1$ that

$$\begin{aligned} &\int_{|x| \leqslant a - t} |u_1(t, x)|^p \, dx \to \infty, \qquad \text{when} \quad t \to a, \text{ if } p\alpha > n, \\ &\int_{|x| \leqslant \varepsilon} |u_1(t, x)|^p \, dx \to \infty, \qquad \text{when} \quad t \to a, \text{ if } p > 0 \text{ and } \beta = 0. \end{aligned} \tag{6.7}$$

Hence if there were a solution $w$ of (6.1) with $w(t, x) = u_1(t, x)$ when $|x| \leqslant a - |t| + \varepsilon$ and $t < a$ it would follow from (6.7) and Sobolev's lemma that

$$\int ||D_x|^\gamma w(t, x)|^2 \, dx \to \infty,$$

$$\text{when} \quad t \to a, \text{ if } \gamma > \frac{n}{2} - \frac{2}{\kappa - 1} \text{ or if } \beta = 0, \gamma > 0. \quad \blacksquare$$

LEMMA 6.5. *Assume that* $0 \leqslant \beta < 1$ *and let* $c_\alpha$ *be as in Lemma* 6.4. *If* $\beta = 0$ *let*

$$f(x) = \frac{c_\alpha}{a^\alpha} \psi(|x|/a), \qquad g(x) = \frac{c_\alpha \alpha}{a^{\alpha+1}} \psi(|x|/a),$$

*where* $\psi \in C_0^\infty$ *satisfies* $\psi(\tau) = 1$, *when* $\tau \leqslant 1 + d$, $d > 0$. *If* $0 < \beta < 1$ *let*

$$f(x) = \frac{c_\alpha (1 - \beta^2)^{\alpha/2}}{(a + \beta x_1)^\alpha} \chi_\beta(x/a), \qquad g(x) = -\frac{1}{\beta} \frac{\partial f(x)}{\partial x_1},$$

*where*

$$\chi_\beta(x) = \chi(\nu(2(1 + x_1) - |x'|^2)) \qquad \nu = \frac{\beta}{1 - \beta}, \qquad \text{and} \qquad x = (x_1, x').$$

*Here* $\chi \in C^\infty$ *satisfies* $\chi(\tau) = 1$ *when* $\tau \geqslant -1/16$ *and* $\chi(\tau) = 0$ *when* $\tau \leqslant -1/8$. *Then*

$$\left( \int ||D_x|^\gamma f(x)|^2 + ||D_x|^{\gamma - 1} g(x)|^2 \, dx \right)^{1/2} \leqslant C_\gamma \frac{(1 - \beta)^{(n+1)/4 - \alpha/2 - \gamma}}{a^{\gamma - (n/2 - \alpha)}}, \tag{6.8}$$

*provided that* $\gamma \geqslant 0$ *if* $\beta = 0$, $0 \leqslant \gamma \leqslant 1$ *and* $\alpha > (n+1)/4$ *if* $0 < \beta < 1$.

*Proof of Lemma* 6.5. If $\beta = 0$ then (6.8) just follows from scaling. If $0 < \beta < 1$ then the estimate for $g$ is just a consequence of the estimate for $f$ since the Riesz transform $|D_x|^{-1} \partial_{x_1}$ maps $L^2$ to $L^2$. By homogeneity, replacing $x$ by $ax$, we may assume that $a = 1$, in which case we can write

$$f(x) = \frac{(1 - \beta^2)^{\alpha/2}}{(1 - \beta)^\alpha} f_0(\nu(1 + x_1), \sqrt{\nu} x'),$$

where

$$f_0(y_1, y') = \frac{c_\alpha}{(1 + y_1)^\alpha} \chi(2y_1 - |y'|^2).$$

Hence

$$\hat{f}(\xi) = \left( \frac{1 + \beta}{1 - \beta} \right)^{\alpha/2} \nu^{-(n+1)/2} \hat{f}_0(\xi_1/\nu, \xi'/\sqrt{\nu}),$$

so a change of variables gives

$$\int (|\xi_1|^2 + \nu\, |\xi'|^2)^\gamma\, |\hat{f}(\xi)|^2\, d\xi$$

$$= (1+\beta)^{\alpha/2}\, \beta^{2(\gamma-(n+1)/4)}\, (1-\beta)^{2((n+1)/4-\alpha/2-\gamma)} \int |\xi|^{2\gamma}\, |\hat{f}_0(\xi_1, \xi')|^2\, d\xi.$$

Hence the remaining case of (6.8) would follow if we could prove that the last integral is convergent. If $\gamma = 0$, then, since the measure of the set $\{y' : 2y_1 - |y'|^2 \geqslant -1/8\}$ is $\leqslant C(1+y_1)^{(n-1)/2}$, we obtain

$$\int |f_0(y_1, y')|^2\, dy_1\, dy' \leqslant C \int_{-1/16}^{\infty} \frac{(1+y_1)^{(n-1)/2}}{(1+y_1)^{2\alpha}}\, dy_1 < \infty,$$

if $\alpha > (n+1)/4$. Similarly, the case $\gamma = 1$ can also be estimated by this quantity so the result follows in general by trivial interpolation. ∎

*Proof of Theorem* 6.2. We want to choose a sequence of data $(f_j, g_j)$ as in Lemma 6.5, with $a = a_j \to 0$, because then (6.2) follows from Lemma 6.4 for some $\gamma < \gamma(\kappa)$. We now want to use Lemma 6.5 to prove that also (6.3) is true. In the easy superconformal case $\kappa \geqslant (n+3)/(n-1)$ then, with $\beta = 0$, we can choose a sequence $a_j \to 0$ for which the right side of (6.8) tends to 0, provided that $\gamma < n/2 - \alpha = n/2 - 2/(\kappa - 1)$. On the other hand, if $\gamma < (n+1)/4 - \alpha/2 = (n+1)/4 - 1/(\kappa - 1)$, then we can choose a sequence $a_j \to 0$ and $\beta_j \to 1$ such that the right side of (6.8) goes to 0, yielding the remaining case. ∎

We now turn to proving the stronger blow-up result which holds for the full range of $\kappa$ if $n = 3$ and for the superconformal range for all $n$. It says that $\|u(t, \cdot)\|_{\dot{\gamma}} \to \infty$ when $t \to T(f, g)$ so that $T^\gamma(f, g) = T(f, g)$. The case of $n = 3$ is special, because it is easier to construct a blow-up solution with certain properties using the positivity of the fundamental solution. However, the geometric techniques we use here are general, so it might be possible to generalize the construction to any dimension. First, in Lemma 6.6, we will need to construct a solution that blows up in a certain way and that has a certain domain of existence.

LEMMA 6.6. *The solution of the ordinary differential equation*

$$f''(t) = (f(t))_+^\kappa, \qquad f(0) = 0, \qquad f'(0) = 1,$$

*satisfies*

$$\frac{ct}{|T_2 - t|^{2/(\kappa-1)}} \leqslant |f(t)| \leqslant \frac{Ct}{|T_2 - t|^{2/(\kappa-1)}}, \qquad \textit{for} \quad 0 \leqslant t < T_2, \tag{6.9}$$

*for some* $0 < T_2 < \infty$. *There are constants* $d_n > 0$ *and functions* $\psi_n \in C_0^\infty$, *depending only on the dimension n, such that the solution* $u_2$ *of* (6.1) *with data* $u_2(0, x) = 0$, $\partial_t u_2(0, x) = \psi_n(|x|)$, *satisfies*

$$\begin{cases} u_2(t, x) = f(t), & \text{when } \ |x| + |t| \leqslant T_2 + d_n,\ t < T_2, \\ u_2(t, x) = 0, & \text{when } \ |x| - |t| \geqslant T_2 + 2d_n, \\ 0 \leqslant u_2(t, x) \leqslant f(t), & \text{for } \ 0 \leqslant t < T_2. \end{cases} \tag{6.10}$$

*It follows that*

$$T(u_2) = T_2, \qquad \text{and} \qquad \int |u_2(t, x)|^p \, dx \to \infty, \qquad \text{as} \quad t \to T_2,\ p > 0. \tag{6.11}$$

*If* $n \leqslant 3$ *and* $2 < \kappa < 3$ *then* $u_3(s, y) = u_2(s - (T_2 + 2d_3), y)$ *is a solution of* (6.1) *in* $\Omega(u_3)$ *such that*

$$\begin{cases} \Omega(u_3) \supset \{(s, y) : |y| \geqslant s\}, & \text{and} \quad T(u_3) = 2T_2 + 2d_3 \\ \Box u_3 = 0, \text{ in } \{(s, y) : |y| \geqslant s\}, & \text{and} \quad u_3 \in C^2(\Omega(u_3)). \end{cases} \tag{6.12}$$

*Moreover, if* $0 < \beta < 1$ *and* $b = b(\beta)$ *is the supremum over all t such that the hyperplane* $\sum_t^\beta = \{(s, y) : s - \beta y_1 = t,\ \ y \in \mathbb{R}^3\} \subset \Omega(u_3)$, *then there is a sequence* $t_j \to b$ *such that*

$$\int |u_3(t_j - \beta y_1, y)|^\kappa \, dy \to \infty. \tag{6.13}$$

*Proof of Lemma* 6.6. It is easy to see that the solution of $f'' = f_+^\kappa$ is implicitly given by

$$\int_0^{f(t)} \frac{dv}{\sqrt{1 + 2\,|v|^{\kappa+1}/(\kappa+1)}} = t,$$

proving that $f(t) \to \infty$, when $t \to T_2$ for some finite $T_2 > 0$. Hence

$$\int_{f(t)}^\infty \frac{dv}{\sqrt{1 + 2\,|v|^{\kappa+1}/(\kappa+1)}} = T_2 - t,$$

from which (6.9) follows. Let $d_n$ be a positive constant to be chosen and take $\psi_n \in C_0^\infty$ satisfying $\psi_n(\tau) = 1$, when $\tau \leqslant T_2 + d_n$, $\psi_n(\tau) \geqslant 0$ and $\psi_n(\tau) = 0$, when $\tau \geqslant T_2 + 2d_n$. If $n \leqslant 3$ then (6.10) follows from a comparison argument using the positivity of the forward fundamental solution of $\Box$ (see e.g. Theorem 3.8 and Lemma 8.9 in [20]). The case $n > 3$ follows from similar reasoning since the fundamental solution is positive in the radial case if

$r-t\geqslant d_n t$ for some $d_n>0$ (see Glassey [7] or Takamura [42]). If $n=3$ it in addition follows from then positivity that $u_2(t,x)\leqslant 0$ and hence $\Box u_2=0$ when $t<0$. This together with (6.10) implies (6.12). That $u_3\in C^2(\Omega(u_3))$ follows from Theorem 5.7.

It only remains to prove (6.13). Let $v(t,r)=|x|\,u_3(t,x)$, where $r=|x|$. A point in $\{(b-\beta y_1, y): y\in\mathbb{R}^3\}\cap\partial\Omega(u)\cap\{(s,y): s>0\}$ always has coordinates $(b-\beta y_1, y_1, 0, 0)$ for some $y_1<-d_3$. Let $(s^\beta, y^\beta)$ be one of these points and let $r^\beta=|y^\beta|$. By Theorem 5.6 $\|u_3\|_{L^q(\Lambda_{s^\beta, y^\beta})}=\infty$. Since $r^\beta>d_3>0$ it follows that there exist a sequence of points

$$\Omega(u_3)\ni(s_j, r_j)\to(s^\beta, r^\beta)\qquad\text{such that}\quad v(s_j, r_j)\to\infty.$$

Since $u_3\in C^2(\Omega(u_3))$ it follows that it is a classical solution and that we can introduce radial coordinates:

$$(\partial_t-\partial_r)(\partial_t+\partial_r)v=r\,|u_3|^\kappa,\qquad v(t,0)=0.$$

Hence integrating this over a characteristic rectangle $R_{t,r}=\{(\tau,\rho): t-r\leqslant \tau+\rho\leqslant t+r,\ 0\leqslant\tau-\rho\leqslant t-r\}$ we obtain

$$v(t,r)=v((t+r)/2, (t+r)/2)-v((t-r)/2, (t-r)/2)$$
$$+\iint_{R(t,r)}|u_3(\tau,\rho)|^\kappa\,\rho\,d\rho\,d\tau.$$

Since $v(s,s)$ is bounded it follows that

$$\iint_{R(s_j, r_j)}|u_3(\tau,\rho)|^\kappa\,\rho\,d\rho\,d\tau\to\infty,\qquad\text{as}\quad j\to\infty.$$

On the other hand, with $t_j=s_j-\beta r_j$, the integral (6.13) becomes, if we introduce polar coordinates,

$$\frac{1}{8\pi}\int_0^\infty\int_{-1}^{1}|u_3(t-\beta\omega_1 r, r)|^\kappa\,d\omega_1\,r^2\,dr=\frac{1}{8\pi\beta}\iint_{B(t_j)}|u_3(\tau,\rho)|^\kappa\,\rho\,d\rho\,d\tau,$$

where $B(t)=\{(\tau,\rho): |\tau-t|\leqslant\beta\rho\}$. If $\beta>0$ it follows that $R(s_j, r_j)\backslash(B(t_j)\cap R(s_j, r_j))$ is bounded away from $\partial\Omega(u_3)$ so $u_3$ is bounded there and hence (6.13) follows. ∎

*Proof of Theorem* 6.3 *in the super conformal case*: $\kappa\geqslant(n+3)/(n-1)$. It follows directly from (6.11) and Sobolev's lemma that $u_2$ satisfies (6.5). Then a simple scaling argument as in the introduction gives the ill-posedness result in Theorem 6.3 for the superconformal range. ∎

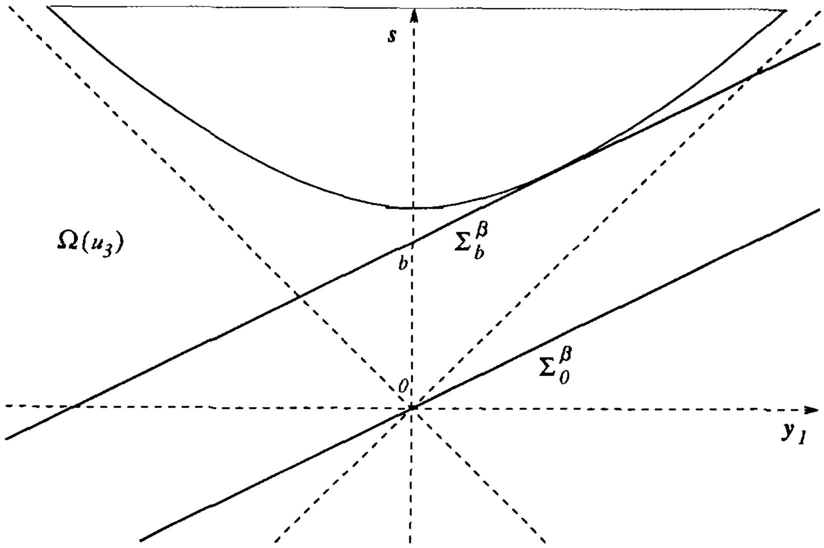



FIG. 6.2. $\Omega(u_3)$: The domain of existence for $u_3(s, y)$.

The rest of this section will be devoted to proving Theorem 6.3 in the subconformal case, $2 < \kappa < 3$, when $n = 3$. Even though this result just concerns $(1+3)$ dimensions, we shall describe a more general geometric method to construct counterexamples as follows. The principle is simple. One starts with a radially symmetric solution of (6.1) with certain properties: it should blow up somewhere inside the light cone $s > |y|$ but it should be regular and decay like a linear solution (as in (6.18)) outside, when $|y| \geqslant s$. This is how the solution $u_3$ in Lemma 6.6 behaves; see (6.12), and Fig. 6.2.

Then one makes a certain combination, (6.14), of a scaling and a Lorentz transformation as described below. If $u$ is a solution of (6.1) then so is

$$\mathscr{S}_a^* u(t, x) = a^{-2/(\kappa-1)} u(t/a, x/a)$$

and $\mathscr{L}_\delta^* u(t, x) = u(s, y)$, where

$$(s, y) = \mathscr{L}_\delta(t, x) = \left(\frac{t - \beta x_1}{\delta}, \frac{x_1 - \beta t}{\delta}, x_2, ..., x_n\right), \qquad \text{and} \qquad \beta = \sqrt{1-\delta^2}.$$

is a Lorentz transformation. Hence, if

$$\mathscr{T}_\delta(t, x) = \left(\frac{t - \beta x_1}{\delta^2}, \frac{x_1 - \beta t}{\delta^2}, \frac{x_2}{\delta}, ..., \frac{x_n}{\delta}\right), \qquad \text{where} \quad \beta = \sqrt{1-\delta^2},$$

then

$$\mathcal{T}_\delta^* u(t,x) = \mathcal{S}_\delta^* \mathcal{L}_\delta^* u(t,x) = \delta^{-2/(\kappa-1)} u(\mathcal{T}_\delta(t,x)) \tag{6.14}$$

is also a solution. We need the following geometric lemma.

LEMMA 6.7. *With the above notation,*

$$\mathcal{T}_\delta(\Sigma_a^0) = \Sigma_a^\beta, \qquad \text{if } \Sigma_a^\beta = \{(s,y) : s - \beta y_1 = a\}, \tag{6.15}$$

*and*

$$\begin{aligned}\mathcal{T}_\delta(\{(t,x) \in \Sigma_0^0 : (x_1-\beta)^2 + x_2^2 + \cdots + x_n^2 \leqslant a^2, \}) \\ = \{(s,y) \in \Sigma_0^\beta : |y| \leqslant s + a\}.\end{aligned} \tag{6.16}$$

*Proof.* The first statement follows directly from the definitions. To prove the second statement, let $X=(t,x)$, $Y=(s,y)$ and let $Q(X,Y) = ts - \langle x, y\rangle$. Set

$$K_{t_0,x_0} = \{(t,x) : Q(X-X_0, X-X_0) = 0,\ t - t_0 \geqslant 0\}, \quad \text{where } X_0 = (t_0, x_0).$$

We then have

$$\begin{aligned}Q(X-X_0, X-X_0) &= Q(\mathcal{L}_\delta X - \mathcal{L}_\delta X_0, \mathcal{L}_\delta X - \mathcal{L}_\delta X_0) \\ &= \delta^4 Q(\mathcal{T}_\delta X - \mathcal{T}_\delta X_0, \mathcal{T}_\delta X - \mathcal{T}_\delta X_0).\end{aligned}$$

Hence $\mathcal{T}_\delta(K_{t_0,x_0}) = K_{s_0,y_0}$, where $(s_0,y_0) = \mathcal{T}_\delta(t_0,x_0)$. If we choose $(t_0,x_0) = (a, \beta a, 0, ..., 0)$ then $(s_0,y_0) = (a, 0, ..., 0)$ and the left hand side of (6.16) is just $K_{t_0,x_0} \cap \Sigma_0^0$ whereas the right hand side is $K_{s_0,y_0} \cap \Sigma_0^\beta$. ∎

By (6.15) $(s,y) = \mathcal{T}_\delta(t,x)$ transforms the hyperplanes $\{(t,x) : t = a\}$ into the hyperplanes $\{(s,y) : s - \beta y_1 = a\}$ so if the domain of existence for $u_3(s,y)$ is as in Fig. 6.2 then the domain of existence for $\mathcal{T}_\delta^* u_3(t,x)$ is as in Fig. 6.3.

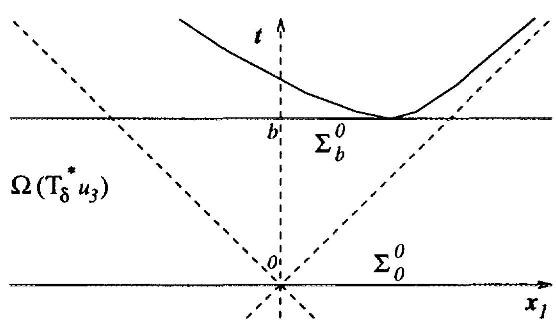



FIG. 6.3. $\Omega(\mathcal{T}_\delta^* u_3)$: the domain of existence for $\mathcal{T}_\delta^* u_3(t,x)$.

With $u_3$ as in Lemma 6.5 we have $u_3 \in C^2(\Omega(u_3))$ and it follows that $\mathcal{T}_\delta^* u_3 \in C^2(\mathcal{T}_\delta(\Omega(u_3)))$. Now, by (6.13) there is a sequence $t_j \to b = T(\mathcal{T}_\delta^* u_3)$ such that

$$\int |\mathcal{T}_\delta^* u_3(t_j, x)|^\kappa \, dx = c_\beta \int |u_3(t_j - \beta y_1, y)|^\kappa \, dy \to \infty.$$

By Sobolev's lemma this implies (6.5) if $\gamma \geqslant 3/2 - 3/\kappa$. Since $3/2 - 3/\kappa < 1 - 1/(\kappa - 1)$ if $2 < \kappa < 3$ this shows that (6.5) holds for some $\gamma < \gamma(\kappa)$. To prove Theorem 6.3 we must now show that

$$\|\mathcal{T}_\delta^* u_3(0, \cdot)\|_{\dot{\gamma}} \to 0, \qquad \text{as} \quad \delta \to 0. \tag{6.17}$$

The first condition in (6.4) can be replaced by $T(f_j, g_j) \leqslant C$, since then by a small additional scaling we can achieve that $T(f_j, g_j) \to 0$ without destroying the other conditions in the theorem. It follows directly from the symmetry and (6.15) that $T(\mathcal{T}_\delta^* u_3) < T(u_3)$. Since our particular solution, $u_3$, is a solution of the linear equation $\Box u_3 = 0$ when $|y| > s$ (6.17) follows directly from (6.21) in Theorem 6.9 below. We will however also include some additional results showing that the same estimate is true even if we do not have a linear solution but only a solution that decays like a linear solution in the region $|y| \geqslant |s|$, see (6.18). This makes our method more general, and in fact one could also use known blow-up and decay results for a solution of $\Box u = |u|^\kappa$ from Lindblad [20] to construct a sequence as above if $2 < \kappa < 1 + \sqrt{2}$. This method also provides another proofs of Theorem 6.2.

THEOREM 6.8. *Assume that $\Omega(u) \supset \{(s, y) : |y| \geqslant s\}$ and that $u$, $u_t$ and $u_{x_i}$, $i = 1, \ldots, n$, satisfy*

$$\begin{cases} |u(s, y)| \leqslant \dfrac{K}{(1 + |s| + |y|)^{(n+1)/4}}, & \textit{when} \quad |y| \geqslant |s|, \\ u(s, y) = 0, & \textit{when} \quad |y| \geqslant |s| + 1. \end{cases} \tag{6.18}$$

*Then*

$$\|\mathcal{T}_\delta^* u(0, \cdot)\|_{\dot{\gamma}} \leqslant CK\delta^{2((n+1)/4 - 1/(\kappa - 1) - \gamma)} \log(1/\delta), \qquad \textit{if} \quad 0 \leqslant \gamma \leqslant 1. \tag{6.19}$$

THEOREM 6.9. *Let*

$$v(s, y) = (2\pi)^{-n} \int e^{-i(s|\eta| - y \cdot \eta)} \hat{g}(\eta) \frac{d\eta}{|\eta|}. \tag{6.20}$$

*Then* $(\widehat{\mathscr{L}_\delta^* v})(0, \xi) = \hat{g}(\eta(\xi))/|\xi|$, *where* $\eta(\xi) = ((\xi_1 - \beta|\xi|)/\delta, \xi_2, \xi_3)$. *In particular if* $v$ *is a radially symmetric solution of* $\square v = 0$ *then*

$$\|\mathscr{T}_\delta^* v(0, \cdot)\|_{\dot\gamma} \leqslant \begin{cases} C\delta^{2((n+1)/4 - 1/(\kappa - 1) - \gamma)} \|v(0, \cdot)\|_{\dot\gamma}, \\ \quad \text{if } \gamma + \dfrac{n-3}{4} > 0, \\ C\delta^{2((n+1)/4 - 1/(\kappa-1) - \gamma)} \log(1/\delta) \|v(0, \cdot)\|_{\dot\gamma}, \\ \quad \text{if } \gamma + \dfrac{n-3}{4} = 0. \end{cases} \tag{6.21}$$

*Proof of Theorem* 6.8. First, we want to show that

$$\|u(\mathscr{T}_\delta(0, \cdot))\|_{L^p(\mathbb{R}^n)} \leqslant CK\delta^{(n+1)/2} \log(1/\delta), \qquad \text{if } 1 \leqslant p \leqslant 2. \tag{6.22}$$

By Lemma 6.7,

$$\begin{aligned} \int |u(\mathscr{T}_\delta(0, x))|^p \, dx &\leqslant C \int_{(x_1 - \beta)^2 + x_2^2 + \cdots + x_n^2 \leqslant 1} \frac{dx}{(|x_1|/\delta^2 + 1)^{p(n+1)/4}} \\ &= C_n \int_{\beta - 1}^{\beta + 1} \frac{dx_1}{(|x_1|/\delta^2 + 1)^{p(n+1)/4}} \int_0^{\sqrt{1 - (x_1 - \beta)^2}} r^{n-2} \, dr \\ &= C_n' \int_{\beta - 1}^{\beta + 1} \frac{(1 - (x_1 - \beta)^2)^{(n-1)/2} \, dx_1}{(|x_1|/\delta^2 + 1)^{p(n+1)/4}}. \end{aligned}$$

Now $1 - (x_1 - \beta)^2 = \delta^2 + 2\beta x_1 - x_1^2$ so after making the change variables $s = x_1/\delta^2$, we see that the above integral can be bounded by

$$\delta^{n+1} \int_{(\beta - 1)/\delta^2}^{(\beta + 1)/\delta^2} \frac{C^2(1 + 2s)^{(n-1)/2} \, ds}{(|s| + 1)^{p(n+1)/4}}.$$

And since this is $O(\delta^{(n+1)/2})$ if $p < 2$ and $O(\delta^{(n+1)/2} \log(1/\delta))$ if $p = 2$, we get (6.22).

We have

$$\begin{aligned} \partial_t u(\mathscr{T}_\delta(t, x)) \pm \partial_{x_1} u(\mathscr{T}_\delta(t, x)) &= \frac{1 \mp \beta}{\delta^2} (u_t(\mathscr{T}_\delta(t, x)) \pm u_{x_1}(\mathscr{T}_\delta(t, x))) \\ \partial_{x_i} u(\mathscr{T}_\delta(t, x)) &= u_{x_i}(\mathscr{T}_\delta(t, x))/\delta, \qquad \text{for } i = 2, ..., n. \end{aligned} \tag{6.23}$$

To prove (6.19) it suffices to prove the result for the special cases where $\gamma = 0$ and $\gamma = 1$, since

$$\|v\|_{\dot\gamma} \leqslant \|v\|_{\dot\gamma_1}^{\theta} \, \|v\|_{\dot\gamma_2}^{(1-\theta)}, \qquad \text{if } \gamma_1 \leqslant \gamma \leqslant \gamma_2 \text{ where } \theta = \frac{\gamma_2 - \gamma}{\gamma_2 - \gamma_1}.$$

First, if $\gamma=0$ then (6.22) directly gives that the $L^2$ norm is bounded, so we only need to estimate the $L^2$ norm of $|D_x|^{-1}\partial_t u$. Since the $L^2$ norm of $|D_x|^{-1}\partial_{x_1}u$ is bounded by the $L^2$ norm of $u$ it follows from (6.23) that we only need to estimate $|D_x|^{-1}(u_t(\mathscr{T}_\delta(t,x))+u_{x_1}(\mathscr{T}_\delta(t,x)))$ which by Sobolev's lemma is bounded by the $L^{2n/(n+2)}$ norm of $u_t(\mathscr{T}_\delta(t,x))+u_{x_1}(\mathscr{T}_\delta(t,x))$, which can be estimated by (6.22). The result for $\gamma=1$ follows from similar arguments. ∎

*Proof of Theorem* 6.9. (6.20) can be written as

$$v(s,y)=(2\pi)^{-n-1}\int e^{-i(s\sigma-y\cdot\eta)}\hat{g}(\eta)\,\delta(\sigma^2-|\eta|^2)\,H(\sigma)\,d\eta\,d\sigma. \tag{6.24}$$

Let $(s,y)=\mathscr{L}_\delta(t,x)$ and $(\sigma,\eta)=\mathscr{L}_\delta(\tau,\xi)$. Then, since everything but $\hat{g}$ is invariant in (6.24), we get

$$(\mathscr{L}_\delta^* v)(t,x)=(2\pi)^{-n-1}\int e^{-i(t\tau-x\cdot\xi)}\hat{g}(\eta(\xi))\,\delta(\tau^2-|\xi|^2)\,H(\tau)\,d\xi\,d\tau,$$

where $\eta(\xi)$ is given by $(|\eta|,\eta)=\mathscr{L}_\delta(|\xi|,\xi)$, since $\tau=|\xi|$ and $\sigma=|\eta|$. Hence $|\eta|=(|\xi|-\beta\xi_1)/\delta$ so

$$\left|\frac{\partial\eta}{\partial\xi}\right|=\left|\frac{1-\beta\xi_1/|\xi|}{\delta}\right|=\frac{|\eta|}{|\xi|}.$$

If we take the inverse Lorentz transformation we obtain $|\xi|=|\tau|=(\sigma+\beta\eta_1)/\delta=(|\eta|+\beta\eta_1)/\delta$ so

$$\int|(\widehat{\mathscr{L}_\delta^* v})(t,\xi)|^2\,|\xi|^{2\gamma}\,d\xi=\int|\hat{g}(\eta)|^2\,|\eta|^{2\gamma}\left(\frac{1+\beta\eta_1/|\eta|}{\delta}\right)^{2\gamma-1}d\eta.$$

If we assume that $g$ is radially symmetric this just becomes

$$\int|\hat{g}(\eta)|^2\,|\eta|^{2\gamma-2}\,d\eta\times C_n\delta^{1-2\gamma}\int_{-1}^{1}(1+\beta\omega_1)^{2\gamma-1}(1-\omega_1^2)^{(n-3)/2}\,d\omega_1.$$

The integral in $\omega_1$ is bounded when $\beta\to1$ if $2\gamma+(n-3)/2>0$, and it is bounded by a constant times $\log((1+\beta)/(1-\beta))\leqslant C\log(1/\delta)$ if $2\gamma+(n-3)/2=0$. ∎

*Remark.* Note that if $u$ satisfies (6.1) or $\Box u=0$ then

$$\int_0^b\int|(\mathscr{T}_\delta^* u)(t,x)|^q\,dx\,dt=\iint_{0\leqslant s-\beta y_1\leqslant b}|u(s,y)|^q\,dy\,ds,$$

$$q=q(\kappa)=(\kappa-1)\,\frac{n+1}{2}.$$

Hence by Theorem 6.9, $v_j = \mathscr{T}^*_{1/j} v$, where $\Box v = 0$ satisfies

$$\|v_j(0, \cdot)\|_{\dot{\gamma}} \to 0 \quad \text{if} \quad \gamma < \gamma(\kappa) \quad \text{and} \quad \int_0^b \int |v_j(t, x)|^q \, dx \, dt \geqslant C > 0.$$

Hence, in a sense, something goes wrong already at the linear level.

## 7. Asymptotic Completeness and Scattering for Small Amplitude Solutions

In this section we are going to prove the scattering and asymptotic completeness results in Theorem 2.2. Recall that we already proved global existence for small norms in Section 4. We start by proving asymptotic completeness. Let $\kappa \geqslant (n+3)/(n-1)$ and $F_\kappa$ be as in Theorem 2.2. Also, let $u$, as in Theorem 2.2, be a global solution to (1.1) with data $(f, g) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$, $\gamma(\kappa) = n/2 - 2/(\kappa - 1)$. Then if the norm of the data is sufficiently small, we need to show that there is data $(f_+, g_+) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$ so that the associated solution to the free wave equation,

$$\begin{cases} \Box u_+ = 0 \\ u_+(0, x) = f_+(x), \qquad \partial_t u_+(0, x) = g_+(x), \end{cases}$$

satisfies

$$\lim_{T \to +\infty} \|u(T, \cdot) - u_+(T, \cdot)\|_{\dot{\gamma}(\kappa)} = 0. \tag{7.1}$$

The key ingredient in the proof of this inequality is (4.31), which says that if the data is small

$$|D_x|^{\gamma(\kappa) - 1/2} F_\kappa(u) \in L^{2(n+1)/(n+3)}(\mathbb{R}^{1+n}). \tag{7.2}$$

As a consequence, we can find an increasing sequence of times, $T_j$, for which

$$\left( \int_{T_j}^\infty \int_{\mathbb{R}^n} \left| |D_x|^{\gamma(\kappa) - 1/2} F_\kappa(u) \right|^{2(n+1)/(n+3)} dx \, dt \right)^{(n+3)/2(n+1)} < 2^{-j}. \tag{7.2'}$$

To exploit this, we let $u_j$ solve the free wave equation with the same data as $u$ at $t = T_j$:

$$\begin{cases} \Box u_j = 0 \\ u_j(T_j, \cdot) = u(T_j, \cdot), \qquad \partial_t u_j(T_j, \cdot) = \partial_t u(T_j, \cdot). \end{cases}$$

Then $u-u_j$ has zero data at $t=T_j$ and satisfies $\Box(u-u_j)=F_\kappa(u)$, $t>T_j$. Hence, (3.11) and (7.2′) give

$$\begin{aligned} \|u(t,\cdot)-u_j(t,\cdot)\|_{\dot{\gamma}(\kappa)} &\leqslant C\left(\int_{T_j}^{t}\int_{\mathbb{R}^n} \left|\,|D_x|^{\gamma(\kappa)-1/2} F_\kappa(u)\right|^{2(n+1)/(n+3)} dx\,dt\right)^{(n+3)/2(n+1)} \\ &< C2^{-j}, \qquad \text{if} \quad t>T_j. \end{aligned} \tag{7.3}$$

Next, notice that if $k>j$ this implies

$$\|u_k(T_k,\cdot)-u_j(T_j,\cdot)\|_{\dot{\gamma}(\kappa)} \leqslant C2^{-j},$$

since $u$ and $u_\kappa$ have the same data at $t=T_k$. Consequently, the energy inequality (3.7) yields

$$\|u_k(0,\cdot)-u_j(0,\cdot)\|_{\dot{\gamma}(\kappa)} \leqslant C2^{-j}, \qquad j<k.$$

Thus, $f_j=u_j(0,\cdot)$, $g_j=\partial_t u_j(0,\cdot)$ is a Cauchy sequence of data in $\dot{H}^{\gamma(\kappa)}\times \dot{H}^{\gamma(\kappa)-1}$. If we let $(f_+,g_+)$ be the limit, then (7.3) and the energy inequality yield (7.1) if, as above, $u_+$ is the solution to the free wave equation with this data. This finishes the proof of asymptotic completeness near the origin.

To prove that the scattering operator exists in a neighborhood of the origin in $\dot{H}^{\gamma(\kappa)}\times\dot{H}^{\gamma(\kappa)-1}$, we must show that, if $(f_-,g_-)\in\dot{H}^{\gamma(\kappa)}\times\dot{H}^{\gamma(\kappa)-1}$ has small norm and if $u_-$ is the solution to the free equation with this data, then there is a solution $u$ to (1.1) satisfying (2.4) and

$$\lim_{T\to-\infty}\|u(T,\cdot)-u_-(T,\cdot)\|_{\dot{\gamma}(\kappa)}=0. \tag{7.4}$$

The proof of this is related to the proof of the existence part of Theorem 2.2. Loosely speaking, the difference is that we are given initial data at $t=-\infty$ instead of at $t=0$. First we note that the mapping $F\to w$ defined as the convolution of $F$ with the forward fundamental solution of $\Box$,

$$w(t,\cdot)=\int_{-\infty}^{t}\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s,\cdot)\,ds, \tag{7.5}$$

also satisfies the estimates (3.8) and (3.11) with now $S_T=(-\infty,T]\times\mathbb{R}^n$. In fact, this follows from (3.8), respectively (3.11), by a limiting procedure, taking $F_j(t,x)=\chi_j(t)F(t,x)$, where $\chi_j(t)\in C_0^\infty$ is equal to 1 when $|t|\leqslant j$.

For the sake of simplicity, let us assume that $1/2\leqslant\gamma(\kappa)\leqslant 3/2$, since, as before, the case corresponding to integer $\kappa$ satisfying $\gamma(\kappa)>3/2$ follows by a straightforward modification of the argument for this case. We now let $u_0 = u_-$ be defined by

$$\begin{cases} \Box u_0(t, x) = 0 \\ u_0(0, x) = f_-(x), \qquad \partial_t u_0(0, x) = g_-(x), \end{cases} \tag{7.6}$$

and define $u_m$, $m = 1, 2, \ldots$ by

$$u_m(t, \cdot\,) = u_0(t, \cdot\,) + \int_{-\infty}^{t} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\kappa(u_{m-1})(s, \cdot\,)\, ds, \tag{7.7}$$

which means that $u_m$ solves $\Box u_m = F_\kappa(u_{m-1})$. Let $M_m(T)$ be as in Lemma 4.3 but with $S_T = (-\infty, T] \times \mathbb{R}^n$. Using the fact that the mapping (7.5) satisfies (3.8) and (3.11) with $S_T = (-\infty, T] \times \mathbb{R}^n$ we can now argue exactly as in the proof of Lemma 4.3 to conclude that (4.22) holds. Also $M_0(T) \leqslant \varepsilon_0$ if the norms in the right hand side of (4.23) are sufficiently small. Now, with $N_m(T)$ defined as in Lemma 4.3 but with $\Lambda_R^- = \{(t, x) \in \mathbb{R}_-^{1+n} : |x| < R - t\}$ we still have (4.22) but we no longer automatically have that $N_0(T) \leqslant C_R M_0(T)$ since the domain now is unbounded. In the conformal case, $\kappa = (n+3)/(n-1)$, $N_0(T) = M_0(T)$, so then the proof of convergence is just as before; see the discussion after Lemma 4.3.

Hence we may assume that $\kappa > (n+3)/(n-1)$. However, if in addition

$$(f_-, g_-) \in \dot H^{1/2} \times \dot H^{-1/2}, \tag{7.8}$$

so that $N_0(T) < \infty$, then, as in the discussion after Lemma 4.3, we see that $u_m$ converges to a solution $u$ of

$$u(t, \cdot\,) = u_0(t, \cdot\,) + \int_{-\infty}^{t} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\kappa(u)(s, \cdot\,)\, ds. \tag{7.9}$$

which satisfies $\|u\|_{L^{2(n+1)/(n-1)}} < \infty$ if we assume that $(f_-, g_-)$ as before also satisfy (2.3). Hence by Corollary 3.4, with $\gamma = 1/2$, respectively, $\gamma = \gamma(\kappa)$:

$$\begin{aligned} \|u(T, \cdot\,)\|_{1/2} &\leqslant C(\|f_-\|_{\dot H^{1/2}(\mathbb{R}^n)} + \|g_-\|_{\dot H^{-1/2}(\mathbb{R}^n)}) \\ \|u(T, \cdot\,)\|_{\gamma} &\leqslant C(\|f_-\|_{\dot H^{\gamma(\kappa)}(\mathbb{R}^n)} + \|g_-\|_{\dot H^{\gamma(\kappa)-1}(\mathbb{R}^n)}). \end{aligned} \tag{7.10}$$

In general, if we only have (2.3) but not (7.8) we can choose a sequence of data $(f_-^k, g_-^k)$ satisfying (2.3) and (7.8) such that $\|f_-^k - f_-\|_{\dot H^{\gamma(\kappa)}(\mathbb{R}^n)} + \|g_-^k - g_-\|_{\dot H^{\gamma(\kappa)-1}(\mathbb{R}^n)} \to 0$. Let now $u = u_k$ be the solution of (7.9), where $u_0 = u_0^k$ is the solution of (7.6) with data $(f_-^k, g_-^k)$. Then we have uniform bounds:

$$\|\,|D_x|^{(n-1)/2-(n+1)/q} u_k\|_{L^{2(n+1)/(n-1)}(S_T)} + \|u_k\|_{L^q(S_T)} + \|u_k(T, \cdot\,)\|_{\gamma} \leqslant C\varepsilon_0, \tag{7.11}$$

for all $T$, where $q=(\kappa-1)/(n+1)/2$. By passing to a subsequence, we may assume that $u_k \to u$ weakly in $\mathscr{S}'(\mathbb{R}^{1+n})$. To conclude that the limit $u$ is a weak solution to (7.9) we must now prove that

$$F_\kappa(u_k) \to F_\kappa(u), \qquad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}). \tag{7.12}$$

Since $\kappa(\gamma)=(n+4-2\gamma)/(n-2\gamma)<1/(1/2-\gamma/n)$, when $\gamma>1/2$, this is a consequence of the following technical lemma:

LEMMA 7.1. *Assume that* $u_k \to u$ *in* $\mathscr{S}'(\mathbb{R}^{1+n})$ *and that for some* $1/2<\gamma<n/2$

$$\|u_k(T,\cdot)\|_{\dot H^{\gamma(\kappa)}(\mathbb{R}^n)} + \|\partial_t u_k(T,\cdot)\|_{\dot H^{\gamma(\kappa)-1}(\mathbb{R}^n)} \leqslant M, \tag{7.13}$$

*where $M$ is independent of $T$ and $k$. Then*

$$u_k \to u, \quad \text{in} \quad L^\kappa_{\mathrm{loc}}(\mathbb{R}^{1+n}), \qquad \text{if} \quad \kappa<1/(1/2-\gamma/n). \tag{7.14}$$

*Proof of Lemma* 7.1. Let $\chi_j(\xi)=\chi(|\xi|/2^j)$, where $\chi \in C_0^\infty(\mathbb{R})$ satisfies $\chi(\xi)=1$ when $|\xi|\leqslant 1$. For a given $\kappa$ let $\gamma_1=n(1/\kappa-1/2)$ so that, by the Hardy-Littlewood-Sobolev inequality and (7.13),

$$\begin{aligned}\|(1-\chi_j(|D_x|))\,u_k(T,\cdot)\|^2_{L^\kappa(\mathbb{R}^n)} &\leqslant C\,\|(1-\chi_j(|D_x|))\,u_k(T,\cdot)\|^2_{\dot H^{\gamma_1}(\mathbb{R}^n)} \\ &\leqslant C\int_{|\xi|\geqslant 2^j} \big|\,|\xi|^{\gamma_1}\,\widehat{u_k}(t,\xi)\big|^2\,d\xi \\ &\leqslant CM^2 2^{-2j(\gamma-\gamma_1)}.\end{aligned} \tag{7.15}$$

Since $u$ satisfies the same estimates, see (4.25), it suffices to show that, if $j$ is fixed, then

$$\chi_j(|D_x|)\,u_k \to \chi_j(|D_x|)u \in L^\kappa_{\mathrm{loc}}(\mathbb{R}^{1+n}), \qquad \text{as} \quad k\to\infty. \tag{7.14$'$}$$

The main step in proving this is to show that

$$|\chi_j(|D_x|)\,u_k(t,x)| + |\nabla_{t,x}\chi_j(|D_x|)\,u_k(t,x)| \leqslant C_j M, \tag{7.16}$$

for some constant $C_j$ independent of $k$. To see the first part, we use (7.13) and the fact that $\gamma<n/2$ to deduce that

$$\begin{aligned}|\chi_j(|D_x|)\,u_k(t,x)|^2 &\leqslant \left(\int_{\mathbb{R}^n} |\chi_j(|\xi|)\,\hat u_k(t,\xi)|\,d\xi\right)^2 \\ &\leqslant \int_{\mathbb{R}^n} |\xi|^{-2\gamma}\,|\chi_j(|\xi|)|^2\,d\xi \cdot \int_{\mathbb{R}^n} |\xi|^{2\gamma}\,|\hat u_k(t,\xi)|^2\,d\xi \leqslant C_j M^2.\end{aligned}$$

Similarly,

$$\begin{aligned}
&|\nabla_{t,x}\chi_j(|D_x|)\,u_k(t,x)|^2\\
&\leqslant \left(\int_{\mathbb{R}^n}\sum_i |\xi_i\chi_j(|\xi|)\,\hat{u}_k(t,\xi)| + |\chi_j(|\xi|)\,\partial_t\hat{u}_k(t,\xi)|\,d\xi\right)^2\\
&\leqslant C\int_{\mathbb{R}^n}|\xi|^{-2(1-\gamma)}\,|\chi_j(|\xi|)|^2\,d\xi\\
&\quad\cdot\int_{\mathbb{R}^n}[\,|\xi|^{2\gamma}\,|\hat{u}_k(t,\xi)|^2 + |\xi|^{2(\gamma-1)}\,|\partial_t\hat{u}_k(t,\xi)|^2]\,d\xi \leqslant C_j M^2,
\end{aligned}$$

finishing the proof of (7.16).

Using (7.16) it is easy to prove (7.14′). If we let $v_k(t,x) = \chi_j(|D_x|)\,u_k(t,x)$ and $v(t,x) = \chi_j(|D_x|)\,u(t,x)$, we claim that $v_k \to v$ uniformly on every compact set $K \subset \mathbb{R}^{1+n}$, which of course is stronger than (7.14′). To see this we first note that $v_k$ must converge weakly to $v$, since $u_k$ converges weakly to $u$. But, on account of (7.16), we can apply the Ascoli theorem to see that $v_k$ must have a subsequence which converges uniformly on $K$, and the limit function must be $v$ in view of the weak convergence. Since the same reasoning implies that every subsequence of $\{v_k\}$ must in turn have a subsequence which converges uniformly to $v$, the claim follows. This finishes the proof. ∎

As we pointed out before, this lemma implies that $u_k$ converges to a weak solution $u$ of (7.9). Since we have the uniform bounds (7.11), it follows that the limit $u$ also satisfies these bounds. (See the discussion after Lemma 4.3). Then since (7.4) follows as in the proof of scattering, we have completed the proof of Theorem 2.2. ∎

*Continuity of Scattering Operator*

In the conformally invariant case where $\gamma = 1/2$, $\kappa = (n+3)/(n-1)$, we can improve Theorem 2.2 and see that the scattering operator $S\colon (f_-, g_-) \to (f_+, g_+)$ is continuous in a neighborhood of the origin.

This would follow if we could show that the map

$$\begin{aligned}
&\{(f,g) \in \dot{H}^{1/2}\times\dot{H}^{-1/2} : \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)} < \varepsilon_0\}\\
&\qquad \ni (f,g) \to (f_+, g_+) \in \dot{H}^{1/2}\times\dot{H}^{-1/2}
\end{aligned}$$

in Theorem 2.2 is continuous. To see this, we note that, if $u_1$ and $u_2$ are two solutions of (2.1), then their difference $w = u_1 - u_2$ satisfies $\Box w = V(u_1, u_2)w$, where by the global existence proof in Section 4, $\|V(u_1,u_2)\|_{L^{(n+1)/2}} < C\varepsilon_0^{\kappa-1}$. Consequently, Theorem 5.1 yields $\|w(T,\cdot)\|_\gamma \leqslant C\,\|w(0,\cdot)\|_\gamma$, $\gamma = 1/2$, which, in turn, leads to the claim.

*Remark.* At present, we cannot prove that the scattering operator is continuous in a neighborhood of the origin in the superconformal case because it seems difficult to establish the continuity of the map

$$\dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1} \ni (f, g) \to |D_x|^{\gamma(\kappa)-1/2} F_\kappa(u) \in L^{2(n+1)/(n-1)}(\mathbb{R}^{1+n}). \tag{7.17}$$

If we could do this, the continuity of the map $\dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1} \ni (f, g) \to (f_+, g_+) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$, and, similarly, the continuity of $S$ near the origin would follow from the above arguments. One way of proving the continuity of (7.17) might be to prove a version of Leibnitz's rule for fractional derivatives for differences (cf. (3.25)), although this would most likely require that $F_\kappa(u)$ be a $C^{1,1}$ function of $u$, and, hence, require that $\kappa \geqslant 2$.

## 8. Radial Symmetry

We shall show here that Theorems 2.1 and 2.2 improve considerably if one assumes that the data $(f, g)$ in (1.1) is radial. Specifically, the next two results show that under the assumption of radial symmetry one gets the results predicted by the simple scaling argument in the introduction for a portion of the subconformal range which increases with the dimension $n$.

Theorem 8.1. *Let $n \geqslant 2$ and set $\gamma_r = 1/2n$. Suppose that*

$$\gamma_r < \gamma(\kappa) = \frac{n}{2} - \frac{2}{\kappa - 1} < \frac{1}{2},$$

*(i.e., $(n^2+4n-1)/(n^2-1) = \kappa_r < \kappa < (n+3)/(n-1)$) and that $(f, g) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$ are radially symmetric. Then there is a $T_* > 0$ and a (weak) solution $u$ to (1.1) verifying*

$$(u, \partial_t u) \in C([0, T_*];\ \dot{H}^{\gamma(\kappa)}(\mathbb{R}^n) \times \dot{H}^{\gamma(\kappa)-1}(\mathbb{R}^n)) \quad \text{and} \quad u \in L^{(\kappa-1)(n+1)/2}(S_{T_*}).$$

Theorem 8.2. *Let $n \geqslant 2$ and $\gamma(\kappa) > \gamma_r$. Then if $(f, g) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$ are radial with norm $< \varepsilon$ (with $\varepsilon > 0$ depending only on $\kappa$, $F_\kappa$, and $n$) there is a unique global (weak) solution to (1.1) satisfying*

$$(u, \partial_t u) \in C_b(\mathbb{R};\ \dot{H}^{\gamma(\kappa)}(\mathbb{R}^n) \times \dot{H}^{\gamma(\kappa)-1}(\mathbb{R}^n)) \quad \text{and} \quad u \in L^{(\kappa-1)(n+1)/2}(\mathbb{R}^{1+n}).$$

*Moreover, for a given $(f, g)$ as above, the there is data $(f_+, g_+) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$ so that the (weak) solution, $u_+$, to the free wave equation with this data satisfies*

$$\lim_{T \to +\infty} \|u(T, \cdot) - u_+(T, \cdot)\|_{\gamma(\kappa)} = 0.$$

*Also, for such $(f, g)$, the map from $(f, g)$ to $(f_+, g_+)$ is continuous in the space of radially symmetric elements of $\dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$. Conversely, if $(f_-, g_-) \in \dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$ is radial and has small norm and if $u_-$ is the solution to the free wave equation with this data, then there is a solution $u$ to (1.1), depending continuously on $(f_-, g_-)$ and satisfying (2.4), for which*

$$\lim_{T \to -\infty} \|u(T, \cdot) - u_-(T, \cdot)\|_{\gamma(\kappa)} = 0.$$

*Thus, the scattering operator $S: (f_-, g_-) \to (f_+, g_+)$ exists and is continuous in a neighborhood of the origin in the space of radial elements of $\dot{H}^{\gamma(\kappa)} \times \dot{H}^{\gamma(\kappa)-1}$.*

One has better results for radial data due to the fact that (3.4) holds for a larger range of exponents. Specifically, if $v$ is the solution to the free wave equation with radial data $(f, g)$, then

$$\|v\|_{L^q(\mathbb{R}^{1+n})} \leqslant C_q \|v(0, \cdot)\|_{\dot{\gamma}}, \qquad \frac{2n}{n-1} < q < \infty, \qquad \gamma = \frac{n}{2} - \frac{n+1}{q}. \tag{8.1}$$

This inequality is well known and follows, for example, from arguments in [19].

In addition to (8.1), we need the following estimates of Harmse [10] for the solution to the inhomogeneous wave equation $\Box w = F$ with zero data:

$$\|w\|_{L^q(\mathbb{R}^{1+n})} \leqslant C_q \|F\|_{L^p(\mathbb{R}^{1+n})}, \qquad \frac{2n}{n-1} < q \leqslant \frac{2(n+1)}{n-1}, \qquad \frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}. \tag{8.2}$$

Also, the proof of (3.4) yields that for $T > 0$

$$\|w(T, \cdot)\|_{\dot{\gamma}} \leqslant C_p \|F\|_{L^p(S_T)}, \qquad \gamma = \frac{n+4}{2} - \frac{n+1}{p}, \qquad 1 < p \leqslant \frac{2(n+1)}{n+3}. \tag{8.3}$$

Unlike (8.1), both (8.2) and (8.3) hold without the assumption of radial symmetry.

If we combine (8.1)–(8.3) we get the following estimates for the solution to the wave equation $\Box u = F$ with data $(f, g)$:

$$\begin{gathered} \|u\|_{L^q(S_T)} + \|u(T, \cdot)\|_{\dot{\gamma}} \leqslant C(\|u(0, \cdot)\|_{\dot{\gamma}} + \|F\|_{L^p(S_T)}), \\ \frac{2n}{n-1} < q \leqslant \frac{2(n+1)}{n-1}, \qquad \frac{1}{p} - \frac{1}{q} = \frac{2}{n+1}, \qquad \gamma = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{q}. \end{gathered} \tag{8.4}$$

Using this improved estimate for radial data one gets Theorems 8.1 and 8.2 using the proof of the existence and scattering results given in the introduction for the conformally invariant case in $(1+3)$-dimensions.

## 9. Further Remarks

It is well known that, for variable coefficients, one also has favorable pointwise estimates for the fundamental solution if the time is small—i.e., the analogue of (3.29). Because of this, locally, all of the estimates in Section 3 hold for uniformly hyperbolic operators with variable coefficients (cf. [2], [15], [33]). Therefore, all of the local existence, uniqueness and propagation results trivially carry over to this setting. A more interesting issue, though, would be to determine when the global results hold.

Another interesting problem would be to determine sharp local existence theorems for quasi-linear hyperbolic operators. Our results, for instance, seem to suggest that when $n \geqslant 5$ the local existence theorems of Klainerman and Machedon [19] and Beals and Bezard [1] concerning quadratic nonlinearities involving $Du$ might be off by half of a derivative.

*Note added in proof.* For further discussions regarding earlier related inequalities, the reader should consult the paper "Generalized Strichartz Inequalities for the Wave Equation" by J. Ginibre and G. Velo, which will appear in the *Journal of Functional Analysis*.

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